Instabilities of a baroclinic, double diffusive frontal zone

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Abstract

The linear theory of double diffusive interleaving is extended to take account of baroclinic effects. I go beyond previous studies by including the possibility of modes with nonzero tilt in the along-front direction, which allows for advection by the baroclinic frontal flow. This requires that the stability equations be solved numerically. My main example is based on observations of interleaving on the lower flank of Meddy Sharon, but a range of parameter values is covered, leading to conclusions that are relevant in a variety of oceanic regimes. The frontal zone is treated as infinitely wide, with uniform gradients of temperature, salinity and alongfront velocity. The stationary, vertically symmetric interleaving mode is shown to have maximum growth rate when its alongfront wavenumber is zero, providing validation for previous studies in which this property was assumed. Besides this, there exist two additional modes of instability: the ageostrophic Eady mode of baroclinic instability and a mode not previously identified. The new mode is oblique (i.e., it tilts in the alongfront direction), vertically asymmetric, and propagating. It is strongly dependent on boundary conditions, and its relevance in the ocean interior is uncertain as a result. Effects of variable diffusivity and buoyancy flux ratio are also considered.

1. Introduction

Thermohaline interleaving is an important mechanism for the lateral mixing of water masses in every ocean basin. In frontal zones, horizontal diffusivities due to interleaving can be comparable with those due to mesoscale eddies, i.e., up to $10^3\text{m}^2/\text{s}$ (Ruddick and Richards, 2003). For example, the recent high phase of the Arctic Oscillation has brought an increased flux of warm, salty, Atlantic water into the Arctic, and that water has mixed with the colder, fresher Arctic water via intrusions of spectacular scale (Rudels et al., 1999). A similar process, though on a smaller scale, has been observed in the Weddell Sea (Robertson et al., 1995). In the Kuroshio-Oyashio confluence, about one half of the water mass transformation needed to create the North Pacific Intermediate Water comes from diffusive interleaving (Talley and Yun, 2001). Other examples include mixing of Red Sea and Persian Gulf waters in the Indian Ocean (Grundlingh, 1985), mixing of Mediterranean water into the North Atlantic (Ruddick, 1992) and mixing across the meridional salinity gradient in the equatorial Pacific (Richards and Edwards,
A purely barotropic front, i.e. one that involves no horizontal density gradient and hence no alongfront current, is a relatively simple system to model mathematically. The linearized equations reduce to algebraic form, which allows analytical determination of the main properties of the initial instability (e.g. Stern, 1967; Toole and Georgi, 1981; McDougall, 1985; Walsh and Ruddick, 1995, 2000). Nonlinear modeling is similarly simplified (Walsh and Ruddick, 1998; Mueller et al., 2006). Unfortunately, attempts to compare the results to ocean observations suffer from chronic uncertainty due to the fact that oceanic fronts generally involve some degree of baroclinicity. Existing work on baroclinic effects (e.g. Kuzmina and Rodionov, 1992; May and Kelley, 1997; Kuzmina and Zhurbas, 2000; May and Kelley, 2002; Kuzmina et al., 2005) relies on simplifying assumptions whose validity is untested. My goal in the present study is to place the current theoretical understanding of baroclinic effects on a sounder footing by computing the linear instabilities of a baroclinic front without the simplifying assumptions of previous studies.

Suppose that the disturbances of interest are sufficiently localized that gradients of background temperature, salinity and alongfront current may be regarded as locally uniform. Linearizing about this stationary state yields a partial differential eigenvalue problem of the form

\[ \sigma \psi = -V(x, z) \frac{\partial \psi}{\partial y} + L \psi. \]  

The solution \( \psi \) is a concatenation of the eigenfunctions for the various perturbation fields, the eigenvalue \( \sigma \) is an exponential growth rate, \( V \) is the alongfront current, \( x, y \) and \( z \) are the cross-front, alongfront and vertical coordinates and \( L \) is a linear, autonomous (having no explicit dependence on space or time) differential operator.

The first term on the right-hand side of (1) represents advection of \( y \)-dependent disturbances by the alongfront flow. If this term is neglected, (1) becomes an autonomous eigenvalue problem which is readily reduced to algebraic form and solved. One case in which the advection term can be neglected is the aforementioned barotropic limit, in which \( V = 0 \).

Even in the baroclinic case \( V \neq 0 \), the non-autonomous advection term in (1) vanishes for cross-front modes, i.e. interleaving modes that tilt only in the cross-front direction, and hence have no dependence on \( y \) (figure 1). The resulting mathematical simplification has led previous investigators of baroclinic interleaving to make this choice (e.g. Kuzmina and Rodionov, 1992; May and Kelley, 1997; Kuzmina and Zhurbas, 2000; May and Kelley, 2002; Kuzmina et al., 2005). May and Kelley (1997) also explored the limit in which baroclinicity is nonzero but small.

Here, I will generalize the theory beyond these approximations via numerical solutions of the full, non-autonomous problem (1). The results will confirm that one interleaving mode is well described by the autonomous model used in previous studies; however, the nonautonomous model supports additional classes of un-
stable modes, including oblique modes (figure 1) as well as the ageostrophic Eady mode of baroclinic instability. These additional modes are strongly dependent on the upper and lower boundaries, and may therefore be less relevant in the ocean interior.

A well-studied example of oceanic interleaving occurred at the edges of Meddy Sharon, a rotating lens of warm, salty water that survived for several years in the cool, fresh North Atlantic before ultimately decaying (Armi et al., 1989; Hebert et al., 1990). Interleaving was identified as the primary mechanism for decay (Ruddick, 1992). In the lower half of the meddy, stratification was such as to support salt fingering instability, and was in this sense representative of midlatitude double-diffusive regimes (You, 2002). The upper half of the meddy supported diffusive convection, which is more common at high latitudes. For this study, I will use the lower flank of Meddy Sharon as my main example, though parameter values have been varied enough to give confidence that the results are more generally valid.

In section 2, I discuss the mathematical model and related theoretical issues. The central results are given in section 3, which concerns the simple case in which all parameterized diffusivities are constants and the salt fingering Schmidt number (ratio of momentum diffusivity to saline diffusivity) is unity. Subsequent sections extend the computations in various ways. In section 4, the effects of varying the Schmidt number are addressed. In sections 5 and 6, different aspects of the parameterized diffusivities are allowed to vary as functions of temperature and salinity gradients. To this point, the background stratification is assumed to be fingering favorable. In section 7, a very different region of parameter space is explored, that in which the background stratification supports diffusive convection. Results are compared with observations in section 8 and are summarized in section 9.

2. Theoretical preliminaries

In this section I will describe the full nonautonomous equations, the autonomous approximation, parameter values and diagnostics used to quantify the physical processes that drive the various instabilities. To begin with, consider a rectilinear frontal zone separating two watermasses having, in general, different temperature, salinity and density. A vertically sheared, along-front current maintains geostrophic balance. Interleaving is treated as a small-amplitude, normal mode perturbation, described by linearized equations of motion.

a. Equations of motion

Motion is assumed to take place on an $f$-plane. Space is measured by the Cartesian co-ordinates $x, y$ and $z$ and the corresponding unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, denoting the cross-front, along-front and vertical directions respectively. The fluid is incompressible and is stratified by two scalars such that the Boussinesq approximation applies. The resulting equations of motion are

\[ \nabla \cdot \vec{u} = 0 \]
\[ \frac{D\vec{u}}{Dt} = -f \hat{k} \times \vec{u} - \nabla p + b \hat{k} + \vec{D}_u \]
\[ \frac{Db_i}{Dt} = D_i. \] \hspace{1cm} (2)

in which $\vec{u}$ is the velocity vector,

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \]

is the material derivative, $t$ is the time, $f$ is the Coriolis parameter, $b$ is the buoyancy $-g(\rho - \rho_0)/\rho_0$, where $\rho$ is the density with characteristic value $\rho_0$ and $g$ is the acceleration due to gravity, and $p$ is the pressure scaled by $\rho_0$. $\vec{D}_u$ and $D_i$ are operators representing small-scale mixing processes; their explicit forms will be given later.

The equation of state is assumed to be linear, so that the net buoyancy $b$ is just the sum of the thermal and saline buoyancies:

\[ b = b_T + b_S. \] \hspace{1cm} (4)

In (2) the subscript $i$ may indicate either $S$ for salinity or $T$ for temperature.

The assumption of vertical periodicity that is natural for the autonomous problem is not appropriate when baro-
clinic advection terms are included. Instead, artificial upper and lower boundaries are located at \( z = \pm H/2 \). They are impermeable, \( w = 0 \), and frictionless, \( \partial u/\partial z = \partial v/\partial z = 0 \). Conditions of constant temperature and salinity, \( b_i = 0 \), are imposed. It will be seen that the boundaries have little effect on interleaving modes, but they destabilize two additional classes of modes which must be carefully distinguished as their oceanic significance is uncertain.

b. Flow decomposition

Motions are assumed to take place in a frontal zone of scale sufficiently large that it can be represented locally by uniform property gradients. The fields are then perturbed by a fully three-dimensional, small-amplitude disturbance. The buoyancy fields are therefore given by

\[
b_i = B_{ix}x + B_{iz}z + \epsilon b'_i(x, y, z, t),
\]

in which \( B_{ix} \) and \( B_{iz} \) are constants, \( i \) once again denotes either \( S \) or \( T \) and the subscripts \( x \) and \( z \) represent partial derivatives. The prime denotes the perturbation and \( \epsilon \) is a small ordering parameter. The background velocity is directed along the front, and varies linearly in the vertical:

\[
\bar{u} = V_z z \hat{z} + \epsilon \bar{u}'(x, y, z, t),
\]

where \( V_z \) is a uniform vertical shear. (Inclusion of a nonzero cross-front shear \( V_x \) will be pursued in a separate study.)

The net vertical gradient of the background buoyancy, \( B_z = B_{Sz} + B_{Tz} \), is equal to the squared buoyancy frequency, commonly written as \( N^2 \). The net horizontal gradient of the background buoyancy determines the background shear via the thermal wind balance: \( fV_z = B_x = B_{Sz} + B_{Tz} \).

The linear perturbation equations are obtained by substituting (5) and (6) into (2-4) and truncating at \( O(\epsilon) \). First, however, I will define the mixing terms \( D_u \) and \( D_i \) explicitly.

c. Parameterized mixing processes

Small-scale fluxes are assumed to be driven by a combination of turbulence and double diffusive instabilities. Following Stern (1967), I assume that small-scale fluxes are vertical and are driven by vertical gradients. When the background stratification supports salt fingering, the saline mixing term is written as

\[
D_S = \frac{\partial}{\partial z} \left[ (K_i + K_S) \frac{\partial b_S}{\partial z} \right].
\]

The term in parentheses is the saline diffusivity. It is the sum of a constant diffusivity \( K_i \) representing ambient turbulence and a diffusivity \( K_S \) driven by double-diffusion. \( K_S \) depends on the density ratio \( R_\rho = -b_{Tz}/b_{Sz} \). Substituting (5) into (7), one obtains

\[
D_S = \epsilon \left\{ [K_i + K_S - R_\rho K'_S] \frac{\partial^2 b'_S}{\partial z^2} - K'_S \frac{\partial^2 b'_T}{\partial z^2} \right\} + O(\epsilon^2),
\]

where \( K'_S \) denotes the derivative of \( K_S \) with respect to \( R_\rho \). All quantities are evaluated at the background value \( \bar{R}_\rho = -B_{Tz}/B_{Sz} \). Note that there is no \( O(1) \) term, i.e. the background state is maintained by balances that do not involve small-scale mixing.

Thermal mixing has a turbulence component that is driven by \( K_i \) in the same way as saline mixing. The double diffusive component is expressed as a thermal buoyancy flux that is proportional to the saline buoyancy flux (Stern, 1967):

\[
D_T = \frac{\partial}{\partial z} \left[ K_i \frac{\partial b_T}{\partial z} \right] + \frac{\partial}{\partial z} \left[ -\gamma K_S \frac{\partial b_S}{\partial z} \right]
\]

\[
= \epsilon \left\{ [K_i + (\gamma K_S)'] \frac{\partial^2 b'_T}{\partial z^2} - [\gamma K_S - R_\rho (\gamma K_S)'] \frac{\partial^2 b'_S}{\partial z^2} \right\} + O(\epsilon^2),
\]

Note that the buoyancy flux ratio \( \gamma \) is allowed to vary with \( R_\rho \).

Momentum mixing is modeled using a viscosity

\[
A = K_i + S\epsilon K_S.
\]
The turbulent component is again just $K_t$, while the double diffusive viscosity is equal to the saline diffusivity times the Schmidt number $Sc(R_p)$. It follows that

$$
\hat{D}_{u} = \frac{\partial}{\partial z} \left[A \frac{\partial \hat{u}}{\partial z} \right]
= \hat{\epsilon} \left\{ \left[ K_t + ScK_S \right] \frac{\partial^2 \hat{w}}{\partial z^2} - \frac{(ScK_S)^2}{B_z} \left[ R_{\nu} \frac{\partial^2 \hat{b}_S}{\partial z^2} + \frac{\partial^2 \hat{b}_T}{\partial z^2} \right] V_z \hat{f} \right\}
+ O(\epsilon^2).
$$

When the background stratification supports diffusive convection, the roles of temperature and salinity are reversed in the sense that temperature obeys a standard flux-gradient relation with diffusivity $K_T(R_p)$ while the saline buoyancy flux is proportional to the thermal buoyancy flux, with proportionality constant $-\gamma B(R_p)$. Moreover, the viscosity due to diffusive convection is expressed as $PrK_T$, where $Pr(R_p)$ is the diffusive convection Prandtl number. All other assumptions are unchanged. Appendix A gives explicit forms for the effective diffusivities that follow from these assumptions.

d. The normal mode perturbation equations

Linearization of the remaining terms in (2-4) is straightforward. The linearized equations are homogeneous in all independent variables except $z$, so that solutions can be sought in the normal mode form

$$u'(x, y, z, t) = \hat{u}(z) \exp[imz + lty + it], \quad (12)$$

Only the real part is physically relevant. The exponential growth rate $\sigma$ is in general complex, while $k$ and $l$ are real cross-front and along-front wavenumbers, respectively, and $\hat{u}(z)$ is a complex vertical structure function.

Substituting (12) and similar expressions for $v'$, $w'$, $b'_S$ and $b'_T$ into the linearized equations yields

$$ik\hat{u} + il\hat{v} + \hat{w}_z = 0, \quad (13)$$

and

$$\sigma \hat{u} = -ilV_z\hat{u} + f\hat{v} - ik\hat{p} + K_{UU}\hat{u}_{zz}, \quad (14)$$

Mixing terms have been written using the effective diffusivities $K_{UU}$, $K_{US}$, $K_{UT}$, $K_{SS}$, $K_{ST}$, $K_{TS}$, and $K_{TT}$. Explicit expressions for these quantities are given in Appendix A.

To obtain a diagnostic equation for the pressure, the first two equations of (14) are multiplied by $ik$ and $il$ respectively, the third is differentiated, and the results added. Using (13), the result is

$$\hat{p}_{zz} - (k^2 + l^2)\hat{p} = -ilf\hat{u} + ikf\hat{v} - 2ilV_z\hat{w} + \hat{b}_z + il(K_{US}\hat{b}_{zz} + K_{UT}\hat{b}_{Tz}). \quad (15)$$

These equations are solved numerically using a Fourier-Galerkin discretization of the $z$ dependence. Details of the solution method are given in Appendix B.

e. The algebraic model for cross-front modes

I now describe the simplified model used in previous studies. Results from this model will be compared with those from the more general model (14). The first term on the right-hand side of each member of (14) describes advection of the perturbation by the sheared alongfront flow. In the special case of modes with $l = 0$, which I refer to here as cross-front modes (figure 1), these terms vanish and (14) is autonomous in $z$. The vertical structure functions then have harmonic form, e.g. $\hat{u} = u_0 \exp(i mz)$, where $u_0$ is a constant and $m$ is the vertical wavenumber. The cross-front slope of the interleaving layer is defined as $s = \tan \theta = -k/m$. The corresponding alongfront
slopes, \( \tan \phi = -1/m \), is zero by assumption. (14) now reduces to an algebraic eigenvalue problem

\[
\sigma \vec{v} = A \vec{v},
\]

where

\[
\vec{v} = \begin{bmatrix} u_0 & v_0 & b_{s0} & b_{T0} \end{bmatrix}^T
\]

and

\[
A = \begin{bmatrix}
  m^2K_{UU} & \frac{-f}{1+\theta^2} & s & \frac{s}{1+\theta^2} \\
  m^2K_{US} & m^2K_{UT} & 0 & \frac{m^2K_{ST}}{f + sV_z} \\
  B_{S_z}(s - s_S) & 0 & m^2K_{SS} & m^2K_{ST} \\
  B_{T_z}(s - s_T) & 0 & m^2K_{TS} & m^2K_{TT}
\end{bmatrix}
\]

Here I have introduced the isohaline slope, \( s_S = -B_{S_z}/B_{S_z} \), and the isothermal slope, \( s_T = -B_{T_z}/B_{T_z} \). The latter is related to the isopycnal slope \( s_p = -B_{y}/B_{z} \) and the isohaline slope by \( s_T = s_p + (s_S - s_p)/R_p \). It is readily shown (e.g. May and Kelley, 1997) that when \( s_p \neq 0 \), the potential energy stored in the total buoyancy field can be released by interleaving motions for which \( s \) lies within the baroclinic wedge \((0, s_p)\). In fingering-favorable stratification, potential energy stored in the salinity field can be released via salt fingering by motions lying within the thermohaline wedge, \((0, s_S)\).

This model for cross-front modes has been studied by Kuzmina and Rodionov (1992), May and Kelley (1997), Kuzmina and Zhurbas (2000) and others. All of these studies assumed uniform diffusivities, i.e. \( K_S' = Sc' = \gamma' = 0 \). For that case, May and Kelley (1997) derived approximate expressions for the fastest-growing mode:

\[
\sigma = \frac{1}{8} \frac{B_z (s_p + \epsilon_z s_S)^2}{|f| 1 + \epsilon_z}, \quad (17)
\]

\[
s = \frac{1}{2} \frac{s_p + \epsilon_z s_S}{1 + \epsilon_z}, \quad (18)
\]

\[
m^2 = \frac{|f|}{K_{UU}}, \quad (19)
\]

where \( \epsilon_z = (1 - \gamma)/(R_p - 1) \). This approximation requires that both the growth rate and the interleaving slope be “small” in various contexts; the reader is referred to May and Kelley (1997) for details.

### f. Parameter values

Parameter values are chosen based on observations of interleaving in Meddy Sharon (Armi et al., 1989; Hebert et al., 1990). Approximation of the meddy boundary by a rectilinear front requires the assumption that motions of interest are much smaller than the scales on which the meddy boundary curves, i.e. \( O(100m) \) in the vertical and \( O(100km) \) in the horizontal. We may also think of these parameter choices as pertaining to a rectilinear front with property gradients similar to those in Meddy Sharon, as could easily exist in other regions of the Mediterranean outflow. The relevant parameter values for the lower and upper flanks of the Meddy Sharon are summarized in Table 2 of May and Kelley (2002), which draws on data from Ruddick (1992).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^3 \times s_S )</td>
<td>27</td>
<td>-8</td>
</tr>
<tr>
<td>( 10^3 \times s_p )</td>
<td>3.6</td>
<td>-1</td>
</tr>
<tr>
<td>( R_p )</td>
<td>1.9</td>
<td>0.34</td>
</tr>
<tr>
<td>( B_z[s^{-2}] )</td>
<td>( 7.4 \times 10^{-6} )</td>
<td>( 9.5 \times 10^{-6} )</td>
</tr>
<tr>
<td>( f[s^{-1}] )</td>
<td>( 7.7 \times 10^{-5} )</td>
<td>( 7.7 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

Table 1: Environmental parameter values describing the lower and upper flanks of Meddy Sharon (May and Kelley, 2002; Ruddick, 1992).

Parameters that represent microscale mixing are derived from various sources. For the main case, all diffusivities are assumed to be constant. In the salt fingering regime, I employ a saline diffusivity \( K_S = 3.0 \times 10^{-5}m^2/s \), which is based on the DNS results of Stern et al. (2001). The flux ratio \( \gamma = 0.59 \) is obtained from linear theory, which has been shown to work well for finite amplitude salt fingers (e.g. Schmitt, 1979; Kunze, 2003). The laboratory experiments of Ruddick et al. (1989) suggest \( Sc = 1 \). In the diffusive convection regime, \( K_T = 3.8 \times 10^{-6}m^2/s \) and \( \gamma_{dc} = 0.17 \) based on the formulas of Kelley (1990). The Prandtl number is 4 based on the observations of Padman (1994).

The vertical domain height is set to \( H = 80m \), much
larger than a typical interleaving scale. (It will be seen that the model delivers additional unstable modes with vertical scales close to $H$. The relevance of those modes for meddy Sharon and other oceanic regimes must be assessed with caution.) Detailed convergence tests have shown that, for all cases considered here, truncation of the Galerkin expansions at $N = 48$ is sufficient to ensure good spatial resolution (see Appendix B).

### g. Kinetic energy and buoyancy flux budgets

Identification of instability mechanisms requires that we differentiate between baroclinic and double diffusive forcing. For vertically periodic interleaving (e.g. May and Kelley, 1997), this is straightforward. Motions lying in the baroclinic wedge (section 2e) are forced by baroclinicity, while those lying in the thermohaline wedge are forced by salt fingering fluxes. To make a more quantitative assessment of instability mechanisms, I employ budgets of kinetic energy and buoyancy flux.

(14) may be combined to give an equation for the perturbation kinetic energy:

$$\sigma_r = \sigma_{Shr} + \sigma_{wS} + \sigma_{wT} + \sigma_\epsilon + \sigma_V,$$

where

$$\sigma_{Shr} = -V_z \langle \hat{\dot{w}}^* \hat{w} \rangle_r / KE,$$

$$\sigma_{wS} = \langle \hat{\dot{w}}^* \hat{b}_S \rangle_r / KE,$$

$$\sigma_{wT} = \langle \hat{\dot{w}}^* \hat{b}_T \rangle_r / KE,$$

$$\sigma_\epsilon = K_{UU} \langle \hat{\dot{w}}^* \hat{r} \hat{z} \rangle_r + \langle \hat{\dot{w}}^* \hat{r} \hat{z} \hat{z} \rangle_r / KE,$$

$$\sigma_V = \langle K_{US} \langle \hat{\dot{w}}^* \hat{b}_S \rangle + K_{UT} \langle \hat{\dot{w}}^* \hat{b}_T \rangle_r \rangle / KE.$$  

and

$$KE = \langle \hat{\dot{u}}^* \hat{u} + \hat{\dot{v}}^* \hat{v} + \hat{\dot{w}}^* \hat{w} \rangle.$$

The subscript $r$ specifies the real part and asterisks denote the complex conjugate. The angle brackets represent the vertical average defined in Appendix B. The first four terms on the right-hand side of (20) represent shear production, saline buoyancy production, thermal buoyancy production and dissipation by parameterized viscosity. The dissipation term is negative definite. The final term, $\sigma_V$, is nonzero only when the parameterized viscosity varies with $R_\rho$. In that case, vertical variations in the double diffusive viscosity due to fluctuations in $b_S$ and $b_T$ (and hence in $R_\rho$) cause convergences and divergences in the vertical flux of background velocity $V$, which in turn generate alongfront velocity fluctuations.

It transpires that both baroclinic and double diffusive interleaving modes are driven primarily by the buoyancy flux $BF = \langle \hat{\dot{w}}^* \hat{b} \rangle_r = (\sigma_{wS} + \sigma_{wT}) \times KE$. As a result, the kinetic energy budget is limited as a tool for discriminating between the two types of instability. I therefore employ an additional budget for the buoyancy flux itself. This is derived from the latter three members of (14), and reads:

$$\sigma_r = \sigma_{acc} + \sigma_{baro} + \sigma_{S} + \sigma_{T},$$

where

$$\sigma_{acc} = \langle \hat{\dot{b}}^* (-\hat{\pi} + \hat{K}_{UU} \hat{w} \hat{z}) \rangle_r / 2BF,$$

$$\sigma_{baro} = -\langle B_z (\hat{\dot{u}}^* \hat{\dot{w}}) + B_z (\hat{\dot{w}}^* \hat{\dot{u}}) \rangle_r / 2BF,$$

$$\sigma_{S} = \langle K_{SS} + K_{TS} \rangle \langle \hat{\dot{w}}^* \hat{b}_S \rangle_r / 2BF,$$

$$\sigma_{T} = \langle K_{ST} + K_{TT} \rangle \langle \hat{\dot{w}}^* \hat{b}_T \rangle_r / 2BF.$$

The first term on the right hand side of (21) represents a correlation between vertical acceleration and the buoyancy perturbation, and does not distinguish between baroclinic and double diffusive forcing. The second term, $\sigma_{baro}$, describes the creation of buoyancy flux by advection of the background buoyancy gradients. This term represents flux creation via baroclinic forcing. For periodic interleaving with $l = 0$, the condition $\sigma_{baro} > 0$ is just the condition that the motions lie within the baroclinic wedge (section 2e). The final pair of terms, $\sigma_{S}$ and $\sigma_{T}$, represent double diffusive forcing. To see this, consider the simple case where ambient turbulence is neglected and the salt fingering diffusivities are constants, so that $K_{ST}$ and $K_{TT}$ are zero and only the saline term is active. If we further assume a vertically periodic perturbation with vertical wavenumber $m$, $\sigma_{S}$ becomes $-m^2 K_S (1 - \gamma) \langle \hat{\dot{w}}^* \hat{b}_S \rangle_r / 2BF$. The flux ratio $\gamma < 1$, and the saline buoyancy flux $\langle \hat{\dot{w}}^* \hat{b}_S \rangle_r$ is negative only for motions lying within the thermohaline wedge. The term is
therefore positive when salt fingering amplifies the buoyancy flux.

3. Results for the case of constant diffusivities, $S_C = 1$

In this section, all diffusivities are assumed to be uniform, i.e. $K_S' = S_C' = \gamma' = 0$. I will begin with the barotropic case, which has been studied previously by Stern (1967), Toole and Georgi (1981), McDougall (1985) and others. The only new factor is the imposition of boundaries. I will then introduce baroclinicity. These results follow on the work of Kuzmina and Rodionov (1992), May and Kelley (1997) and others, but permit new classes of instability by allowing $l$ to be nonzero. This could also be seen as a generalization of the ageostrophic Eady problem (Stone, 1966) to include double diffusive effects.

a. The barotropic limit

The barotropic front exhibits regimes of instability at positive and negative $l$ (figure 2). The modes with $l > 0$ have slightly higher growth rates. Also identifiable is a ridge in the growth rate extending across $l = 0$ from positive $l$. In this subsection I will look in detail at the oblique mode with $l > 0$ and the fastest-growing cross-front mode.

The fastest-growing mode overall is the oblique mode at $(k, l) = (1.71, 8.82) \times 10^{-3} \text{m}^{-1}$ (figure 3; marked “a” on figure 2). This mode has growth rate $5.51 \times 10^{-6} \text{s}^{-1}$, or e-folding time of 2.1 days. As with all modes considered in this paper, this growth rate is more than sufficient for interleaving to grow to finite amplitude by the time it was observed in Meddy Sharon. The vertical structure of this mode is quasiperiodic, with an amplitude envelope that is symmetric about $z = 0$ and decreases at the upper and lower boundaries (figure 3). This mode has $v' \approx 0$ (figure 3b) owing to a state of geostrophic balance that exists between the Coriolis acceleration of $u'$ and the buoyancy force due to alongfront tilt, as was shown for the vertically periodic case by McDougall (1985). As a result, the horizontal component of the interleaving motion is purely cross-front, as shown schematically in figure 1.

For the purpose of quantifying the vertical scales of these quasi-periodic modes, I define the local vertical wavenumber as the vertical derivative of the argument of the structure function, i.e. $\mu(z) = d\phi/dz$, where $\phi$ is defined by $\hat{u} = |u| \exp(i\phi)$. This local wavenumber is then averaged over a layer of thickness $H/10$ centered on
the steering level. [This is the level at which $V(z)$ equals the phase velocity, which is zero for stationary modes.] The resulting average is denoted $\mu_0$, and furnishes a good measure of the vertical scale when the vertical structure is close to periodic, as is true in figure 3. In this case, $\mu_0 = -0.575\text{m}^{-1}$, so that the vertical wavelength is 11m. A characteristic cross-front slope can be identified as $s = -k/\mu_0$, and has the value 0.0030. This slope lies between zero and the isohaline slope, $s_S = 0.027$, i.e. it is within the thermohaline wedge.

Another useful interleaving diagnostic is $r = \max_z |b_T^2|/\max_z |b_S^2|$ the ratio of thermal buoyancy amplitude to saline buoyancy amplitude. This ratio is easily measured in situ, and it varies widely among theoretical interleaving modes. For the barotropic mode shown in figure 3, $r = 1.4$.

The primary source of kinetic energy for this mode is the buoyancy flux driven by vertical advection of the temperature field (figure 3f). This is partly balanced by advection of the salinity field, which requires that work be done against gravity, and by weak dissipation. Shear production ($\sigma_{SR_v}$) and flux convergence due to viscosity fluctuations ($\sigma_Y$) are identically zero.

The net buoyancy flux grows mostly due to the diffusion of salinity by salt fingering (figure 3g). This is consistent with the earlier observation that the cross-front tilt lies within the thermohaline wedge. The vertical acceleration terms amplify the buoyancy flux, while baroclinic buoyancy advection plays a damping role.

The fastest-growing cross-front mode has $k = 6.50 \times 10^{-3}\text{m}^{-1}$ (figure 4; marked “b” on figure 2). Like the oblique mode discussed above, this mode has a symmetric interleaving structure, but with greatly reduced vertical wavelength: $2\pi/|\mu_0| = 3.9\text{m}$. The growth rate is also smaller, with e-folding time 9.9 days. Since $l = 0$, there is no alongfront tilt to balance the Coriolis acceleration. As a result, the alongfront velocity perturbation (figure 4b) is not zero, but is in fact very close to $-\bar{u}$, so that interleaving motions are slanted at about a 45° angle to the cross-front direction, as illustrated in figure 1.

This 45° slant can be predicted using the algebraic model for cross-front modes. With the same approximation used to derive (17-19), one arrives at
\[
\frac{v_0}{u_0} \approx -\frac{f}{|f|} \left(1 - \frac{s_S}{s_B} \frac{\rho}{f^2/B_z} \right).
\] (22)
The slant $v_0/u_0$ is expected to be close to -1 (in the Northern hemisphere) whenever the second term in parentheses is much smaller than one. Here, $s_B = 0$, so the term in question vanishes.

Note also that the buoyancy perturbation is strongly dominated by its thermal component; the amplitude ratio $r$ is 16.8. This property of the cross-front mode also follows from the algebraic model. Eliminating $u_0$ from the third and fourth equations of (16) and assuming constant diffusivities (so that $K_{ST} = K_{TT} = 0$ and $K_{TS} = -\gamma K_{SS}$), one obtains
\[
r = R_\ell + (R_\ell - \gamma) \frac{m^2 K_{SS}}{\sigma}.
\] (23)
Here, $R_\ell = R_\rho(s - s_T)/(s - s_S)$ is the along-intrusion density ratio (May and Kelley, 2002). Both $R_\ell$ and $\gamma$ are typically of order unity; here $R_\ell = 1.0$ and $\gamma = 0.6$. The ratio $m^2 K_{SS}/\sigma$ was shown by May and Kelley (1997) to be $\gg 1$ over a wide range of parameter space. It is therefore to be expected that $r \gg 1$ for this mode.

Like the oblique mode, this cross-front mode is driven mainly by the thermal buoyancy flux (figure 4f). Unlike
the oblique mode, this mode does very little work in advecting the salinity field. Instead, the thermal buoyancy flux is balanced mainly by strong dissipation, which is enhanced due to the reduced vertical scale. The buoyancy flux budget (figure 4g) is very similar to that found for the oblique mode. The buoyancy flux is driven by salinity diffusion and vertical acceleration, and opposed by baroclinicity. This is consistent with the fact that the characteristic slope, $s = -k/\mu_0 = 0.0041$, lies within the thermohaline wedge.

**b. Weak baroclinicity**

Before introducing baroclinicity at the level observed in Meddy Sharon, I first show the effect of very weak baroclinicity on the fastest-growing mode of the barotropic limit (figure 3). The isopycnal slope is set to $s_\rho = 1 \times 10^{-5}$, which is more than two orders of magnitude less than the observed value and three orders of magnitude less than the isohaline slope. The eigenfunctions for the fastest growing mode are shown in figure 5. The horizontal wavenumbers $k$ and $l$ are virtually unchanged from the barotropic limit, as is the real growth rate $\sigma_r$. The growth rate has acquired a small imaginary part, however, which is associated with a dramatic change in the physical structure of the mode. While the mode retains the vertically quasiperiodic character of an interleaving instability, the amplitude envelope is now shifted. The oscillation frequency $\sigma_i$ corresponds to alongfront propagation at a velocity equal to that of the background flow at the steering level, $z_{St} = -\sigma_i/lV_z = 16.6$ m. As is evident upon taking the complex conjugate of (14) and replacing $z$ with $-z$, oblique modes appear in antisymmetric, conjugate pairs, so that this mode is accompanied by another of equal growth rate focused below the midplane.

**c. Observed baroclinicity**

I now increase the isopycnal slope to the observed value, $s_\rho = 0.0036$. This has a significant effect on the stability properties of the front, as shown in figure 6 (cf. figure 2). Three distinct classes of instability are evident. Still present are the regions of oblique instability at positive and negative $l$, but these modes have changed from stationary to oscillatory, as indicated by the dotted contours. The ridge of positive growth rate extending across $l = 0$, evident in figure 2, is now interrupted by a narrow strip in which stationary instability is apparent. These two classes of modes will be seen to be continuations of the oblique and cross-front modes seen in the barotropic case. Finally, stationary instability also appears in a pair of small regions that are symmetric about the $l$ axis at small $k$. This is the ageostrophic Eady mode of baroclinic instability.

In addition to examining the spatial structures of the modes shown on figure 6, we can obtain a more general view of their dependence on baroclinicity and their relationship with the modes found in the barotropic limit by examining the growth rate, steering level and wavenumbers of the fastest-growing mode of each class as functions of the isopycnal slope $s_\rho$. These functions are shown in figure 7. The vertical line in figure 7d indicates the observed isopycnal slope. Because the $s_\rho$ axis is logarithmic, pertinent results for the barotropic limit $s_\rho = 0$ are shown by crosses and circles.

The oblique mode (thick, solid curves on figure 7) stabilizes very slightly with increasing $s_\rho$ until about $s_\rho = 10^{-3}$, beyond which its growth rate increases dramatically. This mode also exhibits rapid increases in both
components of the preferred wavenumber at the same isopycnal slope. As $s_\rho$ is reduced, all properties of the oblique mode approach the values found in the barotropic case $s_\rho = 0$ (crosses on figure 7), although the steering level (figure 7b) remains significantly far from zero at $s_\rho = 10^{-5}$, the lowest isopycnal slope plotted. This shows once again that even very weak baroclinicity is able to break the vertical symmetry of the interleaving mode.

Eigenfunctions of the oblique mode (figure 8) show a structure and dynamic that is dramatically different from the barotropic case in some respects yet very similar in others. The mode propagates to the right, with phase velocity equal to that of the background flow at the steering level. The steering level is now close to the upper boundary. The fact that the steering level approaches the boundary but does not go beyond it (in this or any other mode found in this study) suggests that a semicircle theorem (e.g. Howard, 1961; Pedlosky, 1979) may be implicit in the present mathematical model. The vertical structure is no longer periodic (suggestive of interleaving) but is instead strongly asymmetric, with energy concentrated in the vicinity of the steering level. Each right-going mode like that shown in figure 8 is accompanied by a left-going mode growing at the same rate and concentrated near the

Figure 6: Stability diagram $\sigma_r(k,l)$ for the baroclinic case $s_\rho = 3.6 \times 10^{-3}$. Solid (dotted) contours represent stationary (propagating) instability. Regions of large growth rate are highlighted by dark shading. Letters represent particular cases to be discussed in detail.

Figure 7: Properties of the fastest-growing members of various mode classes as functions of the isopycnal slope. (a) growth rate, (b) steering level, (c) cross-front wavenumber, (d) alongfront wavenumber. “CF” (thick, gray curve) represents the stationary cross-front mode. The May and Kelley (1997) approximation (17) is shown by the dotted curve. Circles at the left-hand side of the plot represent the properties of the corresponding mode in the barotropic limit $s_\rho = 0$. Thick, solid curves represent the oblique mode, and crosses its properties at $s_\rho = 0$. Thin solid curves represent the Eady mode, and thin dashed curves show the corresponding values in the quasigeostrophic approximation. The vertical line on (d) indicates the observed isopycnal slope.
lower boundary. As in the barotropic case, alongfront velocities are very weak in comparison to across-front velocities.

The budgets (figure 8f,g) are similar to those found in the barotropic case (figure 3f,g), and show that the mode is weakly dissipative and driven by buoyancy fluxes that originate in the parameterized mixing due to salt fingers. The shear production term is negligible, even though baroclinic shear is nonzero and baroclinic effects are evident in other ways. In the buoyancy flux budget, the baroclinic term is negative, indicating that baroclinicity acts to oppose interleaving growth.

The validity of the present model for strongly asymmetric modes such as that shown in figure 8 is limited due to the effect of the artificial boundaries. Imposition of more realistic boundary conditions (e.g. Fukamachi et al., 1995) complicates the problem significantly by rendering the stability equations non-autonomous in $x$ as well as $z$, and is therefore deferred to a future study.

Like the oblique mode, the cross-front mode is a continuous extension of its counterpart in the barotropic case (figure 7, thick gray curves, circles), and its growth rate and cross-front wavenumber increase above $s_\rho = 10^{-3}$. Unlike the oblique mode, the cross-front mode retains its stationary character and its vertically-symmetric interleaving structure in the presence of baroclinicity (figure 9; cf. figure 4). The slant of the interleaving motions in the $x-y$ plane is virtually unchanged: about $45^\circ$ to the right of the cross-front orientation, and the saline buoyancy perturbation remains considerably smaller than its thermal counterpart ($r = 13.6$). The kinetic energy and buoyancy flux budgets again show a very similar structure to the barotropic case. The buoyancy flux is almost purely thermal in origin and is opposed mainly by viscous dissipation. The buoyancy flux is driven mainly by double diffusive effects. This is consistent with the fact that the cross-front slope $s = 0.0054$ is within the thermohaline wedge $(0, 0.027)$ but is outside the baroclinic wedge $(0, 0.0036)$.

The cross-front mode has $\mu_0 = -1.61 m^{-1}$, or vertical wavelength $3.9 m$. As is discussed in detail below, this value is considerably smaller than the wavelength of interleaving observed on the lower flank of Meddy Sharon, which was approximately $25 m$ (Ruddick, 1992). The growth rate and cross-front wavenumber of the cross-front mode are approximated very accurately by (17-19), despite the imposition of upper and lower boundaries in the numerical model and the assumptions made in derivation of the analytical approximation (figure 7, compare thick gray and thin dotted curves).

Figure 8: Eigenfunctions and budgets for the fastest growing oblique mode in the baroclinic case $s_\rho = 3.6 \times 10^{-3}$. Solid (dashed) curves represent the real (imaginary) part. The wavenumber is marked by the symbol “a” on figure 6.

Figure 9: Eigenfunctions and budgets for the fastest growing cross-front mode in the baroclinic case $s_\rho = 3.6 \times 10^{-3}$. Solid (dashed) curves represent the real (imaginary) part. The wavenumber is marked by the symbol “b” on figure 6.
Figure 10: Eigenfunctions and budgets for the Eady mode. Solid (dashed) curves represent the real (imaginary) part. The wavenumber is marked by the symbol “c” on figure 6.

Figure 10 shows eigenfunctions and budgets for the stationary mode centered near \( k = 0 \), \( l = 7 \times 10^{-4} \text{m}^{-1} \) on figure 6, the ageostrophic Eady mode. The vertical structure is broad, with significant signal near the upper and lower boundaries. Both its growth rate and its alongfront wavenumber are reasonably well approximated by quasi-geostrophic theory [e.g. Pedlosky (1979); thin, dashed curves on figure 7]. The kinetic energy budget (figure 10f) is dominated by a near-balance between strong thermal and saline buoyancy fluxes, with negligible contributions from shear production and viscous dissipation. In contrast to the double diffusive modes discussed above, the buoyancy flux is driven primarily by the baroclinic term (figure 10g).

**d. Strong baroclinicity**

A final case of interest is baroclinic interleaving. The potential for baroclinicity to amplify interleaving has been noted by May and Kelley (1997), although the degree of amplification was not quantified in that study. As shown in figure 7, growth rates of all modes increase for \( s_\rho > 10^{-3} \). Figure 11 shows eigenfunctions of the fastest-growing interleaving mode for the case \( s_\rho = 10^{-2} \), i.e. the isopycnal slope exceeds the observed value by about a factor of three. The mode has slope 0.0077, which is within both the thermohaline wedge (0, 0.027) and the baroclinic wedge (0, 0.010).

In contrast to the observed case \( s_\rho = 3.6 \times 10^{-3} \), shown in figure 9), the shear production is no longer negligible but is slightly negative, i.e. interleaving reinforces the background shear. The buoyancy flux budget is dominated by the advective term, but note that the baroclinic term has changed from negative to positive, i.e. baroclinic energy conversion acts to accelerate interleaving. The salt fingering term, however, is considerably larger than the baroclinic term. This indicates that, at least in terms of the buoyancy flux, double diffusion remains the primary driving force behind interleaving at this level of baroclinicity. If \( s_\rho \) is increased further, \( \sigma_{\text{baro}} \) becomes larger than \( \sigma_S \); the two terms are equal when \( s_\rho = 0.017 \) (about 0.6 times the isohaline slope).

**e. Discussion**

I have identified three classes of instability: the cross-front, oblique and Eady modes. Interleaving motions observed in Meddy Sharon are qualitatively identifiable with the cross-front mode, even though it is not the fastest growing mode in this model. The fastest-growing mode is the oblique mode (figure 8), which is strongly boundary-trapped. As there are no boundaries in the ocean interior,
the relevance of this mode for Meddy Sharon is not clear. It is possible that a layer of strong stratification could act as a partial boundary, allowing the oblique mode to exist, but it may also be that the oblique mode is entirely stabilized in the ocean interior.

The Eady mode is also strongly dependent on boundaries. A convincing analysis of baroclinic instability in this context would have to account for the ellipsoidal geometry of the meddy. In fact, some meddies may undergo baroclinic instability early in their life cycles (Beckmann and Case, 1989), but there is no indication that Meddy Sharon was baroclinically unstable when the observations of Armi et al. (1989) were made.

Even if baroclinic instability were present, the spatial scales of the Eady mode are so much larger than those of cross-front interleaving that the two instabilities could coexist. The potential for coexistence is even clearer in the case of the oblique modes, whose amplitude is negligible throughout most of the vertical domain (figure 8) where the cross-front interleaving mode is focused (figure 9).

The most important conclusion we can reach about the interleaving mode shown on figures 4, 9 and 11 is that it is indeed a cross-front mode, i.e. its growth rate is maximized at $l = 0$. More precisely, the fastest-growing interleaving mode for the Meddy Sharon lower flank parameter set with $H = 80\text{m}$ has $l = 1.3 \times 10^{-7}\text{m}^{-1}$, so that the alongfront wavelength is of the order of the Earth’s circumference. As $H$ is increased further, $l$ vanishes to machine precision. As a result, use of the algebraic model described in section 1e is justified. This provides retroactive validation of the assumptions underlying Kuzmina and Rodionov (1992), May and Kelley (1997) and other previous studies of baroclinic effects, and also lays a foundation for further investigations based on the algebraic model.

To assess the general validity of this result, however, I must relax some of the assumptions that have been made in this section. I will begin by varying the double diffusive Schmidt number, which has been set to unity so far. I will then relax the assumptions that the diffusivity $K_S$ and the flux ratio $\gamma$ are constants, and instead allow them to vary as functions of $R_\rho$. Finally, I will alter the parameter regime drastically by considering the diffusively unstable interleaving observed in the upper half of the Meddy Sharon. All of these cases present significant points of interest, but the main goal will be to test the assumption that the fastest-growing interleaving mode has $l = 0$, as that justifies the use of the algebraic model for further investigations.

4. Dependence on the Schmidt number

The Schmidt number of salt fingering plays a crucial role in governing the vertical scale of interleaving layers (Toole and Georgi, 1981). In linear interleaving models (e.g. Walsh and Ruddick, 1995; May and Kelley, 1997), it is often found that $Sc$ must be chosen greater than one in order to reproduce the vertical scale of observed interleaving layers. While early experiments on salt fingers in shear (Ruddick, 1985) suggested that $Sc > 1$, more recent evidence indicates that $Sc$ is of order one or even smaller (Ruddick et al., 1989; Smyth and Kimura, 2007; Kimura and Smyth, 2007). These studies involve relatively simple, unidirectional shear flows. It is possible that in the veering currents of baroclinic interleaving, momentum could be fluxed more effectively, justifying the assumption $Sc > 1$.

Here, I look briefly at the case $Sc = 4$ (figure 12). All other parameter values are unchanged from section 3. The isopycnal slope has the observed value 0.0036. The most important result is that the cross-front interleaving mode, with maximum growth rate at $l = 0$, is still evident. It is clear from comparison with figure 6 that growth rates of both oblique and cross-front interleaving modes are maximized at reduced horizontal wavenumbers.

The vertical structure of the cross-front mode (figure 13) shows a twofold decrease in wavenumber $\mu$, from 1.61 to 0.83. This is consistent with the square root dependence predicted by (19). The vertical wavelength is now 7.6m, still considerably smaller than the observed value, 25m
Figure 12: Stability diagram $\sigma_r(k,l)$ for the baroclinic case with $Sc = 4$. The isopycnal slope has the observed value 0.0036; all other parameters are as in section 3. Solid (dotted) contours represent stationary (propagating) instability. Regions of large growth rate are highlighted by dark shading.

Figure 13: Eigenfunctions and budgets for the fastest growing cross-front mode for the case $Sc = 4$, indicated by the symbol “a” on figure 12. Solid (dashed) curves indicate real (imaginary) parts.

Figure 14: Stability diagram $\sigma_r(k,l)$ for the baroclinic case with variable diffusivity $n_f = 2$. All other parameters are as in section 3. Solid (dotted) contours represent stationary (propagating) instability. Regions of large growth rate are highlighted by dark shading. Letters represent particular cases to be discussed in detail below.

### 5. Variable diffusivities

Walsh and Ruddick (1998) employed a simple parameterization for the saline diffusivity of salt fingers: $K_S = K_{S0}R_\rho^{-n_f}$. Here, I repeat the analysis of section 3c using this parameterization. A good fit to the direct simulations of Stern et al. (2001) is achieved when $n_f = 2$. To facilitate comparison with results given in section 3, the constant $K_{S0}$ is chosen so that $K_S = 3 \times 10^{-5}$ m$^2$/s when $R_\rho = 1.9$. 

The growth rate of the Eady mode is negligibly affected when $Sc = 4$ (as expected due to the mode’s inviscid character), but the growth rate of the fastest-growing oblique mode is reduced by a factor two. Despite the damping effect of increased $Sc$, the oblique mode remains the fastest-growing mode overall.
Salt fingering, which arise due to perturbations in \( R_g \), is amplified by convergences in the flux of background alongfront velocity by 15a,b). The alongfront component is strongly saline and (negative) thermal terms. The strongly negative thermal term is a consequence of variable diffusivity (21,24). Through it, temperature fluctuations reduce the buoyancy flux indirectly via their effect on \( R_p \) and hence on the local strength of double diffusion. This mechanism damps the buoyancy flux that drives interleaving, and therefore accounts for the reduced growth rate. 

The saline buoyancy perturbation remains much smaller its the thermal counterpart, though the difference is smaller than in the previous case (figure 15d,e). The amplitude ratio \( r \) is reduced to 2.7. Once again, growth of the interleaving mode is maximized when \( l = 0 \).

6. Variable flux ratio

For the barotropic, nonrotating case, Walsh and Ruddick (2000) showed that the tendency for the flux ratio \( \gamma \) to decrease with increasing \( R_p \) causes the cross-front interleaving instability to exhibit “ultraviolet catastrophe” (UVC) behavior. i.e. its growth rate increases monotonically with increasing vertical wavenumber rather than being maximized at some preferred scale. This result is profoundly inconsistent with observations, which reveal a well-defined vertical scale. It would therefore be interesting to know if the UVC persists in the baroclinic case. To this end, I repeat the previous analysis with constant diffusivities, allowing \( \gamma \) to vary in accordance with the approximation \( \gamma = R_p - \sqrt{R_p(R_p - 1)} \) (Stern, 1975). Other parameter values are as in section 3c.

The UVC is not directly accessible in this calculation due to finite vertical resolution. However, as resolution is increased, the cross-front mode increases with no apparent limit, indicating that the UVC also exists in the baroclinic case. While the present numerical approach cannot establish the existence of UVC rigorously, as can be (and has been) done for cross-front modes using the algebraic model, it can give a useful indication of the properties of oblique modes. It can also confirm that the growth rate is indeed maximized at \( l = 0 \), so that use of the algebraic model for more detailed study of the UVC is justified.

Bordering the cross-front mode is a region of oblique instability that represents an extension of the cross-front mode to nonzero \( l \) (figure 16). The real part of the growth rate is continuous between the two domains. As \( l \) passes a critical value \( l_c \), the imaginary part of the growth rate becomes nonzero, and the vertical structure assumes the asymmetric form characteristic of the oblique modes discussed previously (figures 5, 8). As resolution increases,
the cross-front mode dominates over the oblique modes that exist for uniform $\gamma$.

The fastest-growing stationary mode in the strip surrounding $l = 0$ has $k = 8.7 \times 10^{-3} \text{m}^{-1}$ and $|l| < 5 \times 10^{-6} \text{m}^{-1}$, i.e. $l = 0$ for all practical purposes. The critical alongfront wavenumber $l_c$, such that the mode is stationary for $|l| < l_c$, has the value $7 \times 10^{-5} \text{m}^{-1}$ when $N = 48$, but increases to $9 \times 10^{-5} \text{m}^{-1}$ when $N = 64$ (figure 16, lower panel). Extrapolation from these two cases suggests that, as resolution increases, the band of stationary instability becomes wider in $l$, the growth rate increases, and the growth rate continues to be maximized at $l = 0$.

a. The effect of ambient turbulence

Walsh and Ruddick (2000) showed that the UVC can be suppressed by ambient turbulence in the barotropic, non-rotating case. It would be interesting to know if the same result obtains in the baroclinic case. That issue can be addressed most effectively using the algebraic model, provided that the assumption $l = 0$ remains valid. Here, I check that assumption by introducing a small turbulent diffusivity into a model with finite vertical resolution ($N = 48$).

As $K_t$ is increased from zero to $1 \times 10^{-6} \text{m}^2/\text{s}$ (figure 17, dashed curve) and again to $2 \times 10^{-6} \text{m}^2/\text{s}$ (dotted curve), the growth rate of the fastest-growing interleaving mode is strongly damped. Nevertheless, the growth rate is maximized at $l = 0$ in every case. Assuming that this result remains valid at higher resolution (including the effectively infinite resolution of the algebraic model) and at all values of $K_t$, I conclude that further study of the UVC using the algebraic model is justified. The results will be described in a separate publication.

7. The diffusive convection regime

The final parameter regime tested is one in which the background stratification supports diffusive convection rather than salt fingering: the upper flank of Meddy Sharon. Because I retain the definition of $R_p$ that is normally used in salt fingering studies, viz. $R_p = -b_{Tz}/b_{Sz}$, revised forms for the equivalent diffusivities ($K_{UU}$, $K_{SS}$, etc.) are needed. These are given in appendix A.

The general form of the stability diagram (figure 18) is remarkably similar to the salt fingering case, given the dramatic difference in parameter regime. The oscillatory interleaving mode is evident, with maximum growth rate on the $l > 0$ lobe. The Eady mode is visible at small
$k$. Most importantly for the present considerations, the symmetric interleaving mode is visible in a thin strip surrounding $l = 0$, with maximum growth rate at $l = 0$. The fastest growing mode has $k = 6.17 \times 10^{-3} \text{m}^{-1}$ and growth rate $\sigma = 9.26 \times 10^{-7} \text{s}^{-1}$, or e-folding time 12 days. The dominant vertical scale is 3.5m, considerably smaller than the observed vertical scale (section 8). The cross-front tilt of the interleaving plane is -0.0034.

As one would expect, the physics of the instability is in some respects opposite to the salt fingering case (figure 19). The buoyancy perturbation is dominated by its saline (rather than its thermal) component, to the extent that $r$ is reduced to 0.12. Correspondingly, kinetic energy for interleaving motions is supplied by the saline (as opposed to thermal) buoyancy flux. The primary source of this buoyancy flux is double diffusive dissipation of thermal buoyancy anomalies, $\sigma_T$.

### 8. Comparison with observations

The cross-front slope $s = -k/\mu$, vertical wavelength $h = 2\pi/\mu$ and amplitude ratio $r = \max_z |b'_z|/\max_z |b'_l|$ of the interleaving layers observed in Meddy Sharon provide useful standards against which to compare theoretical results. In these definitions, $\mu$ represents either the vertical wavenumber $m$ for periodic interleaving or the equivalent quantity $\mu_0$, defined for quasi-periodic interleaving. Interleaving slopes given in table 2 are tabulated in May and Kelley (2002) based on observations described in Ruddick (1992). Vertical wavelengths represent the spectral peaks shown in figure 14 of Ruddick (1992), with an estimated uncertainty of 20%. Observed values of $r$ were obtained from figures 10 and 12 of Armi et al. (1989). Table 2 gives these values, along with corresponding values for various unstable modes computed above.

I look first at the cross-front slope, $s$. In the case of the barotropic oblique mode discussed in section 3a, interleaving layers slope far more gently than the observed layers. In contrast, all of the cross-front modes have slopes within error of the observations. This close agreement between observed and computed slopes echoes the findings of May and Kelley (2002).

In terms of vertical wavelength, no computed mode compares well with the observations. The barotropic oblique mode has wavelength 11m, about half of that observed, whereas the cross-front modes typically have wavelength around 4m. This discrepancy may be remedied by assuming a larger value for the salt fingering Schmidt number. For example, May and Kelley (1997) obtained vertical wavelengths within error of the obser-
vations by using \( Sc > 1 \). To match the observation exactly would require \( Sc = (25/4)^2 = 39 \). There is no corroborating evidence for values this high. In the veering shear flow that characterizes baroclinic interleaving, \( Sc \) has not been measured and could possibly be \( \gg 1 \).

In the barotropic limit, however, the shear is parallel, and the value of \( Sc \) in a parallel shear flow is \( O(1) \) or even smaller (Kimura and Smyth, manuscript in preparation).

It could be that some other mechanism that fluxes momentum more effectively than scalars is responsible for the relatively large vertical wavelength of observed interleaving layers. One possibility is gravity waves excited via the collective instability of salt fingers (e.g. Stern, 1969; Stern et al., 2001). Another possible explanation is that interleaving layers undergo a subharmonic instability at finite amplitude, causing their vertical wavelength to increase.

Another intriguing discrepancy is in the amplitude ratio \( r \). This ratio is observed to be of order unity on both the upper and lower flanks of Meddy Sharon. The barotropic oblique mode has \( r \sim 1 \). For the cross-front modes of baroclinic fronts, \( r \) is of order 10 in the fingering-favorable stratification of the lower flank and of order \( 10^{-1} \) on the upper flank. Only when the saline diffusivity is allowed to vary with \( R_{\rho} \) (section 5) does the cross-front mode exhibit \( r \) within error of the observations.

It should be noted that the foregoing comparisons do not include the case in which \( \gamma \) varies with \( R_{\rho} \), because this case leads to UVC and is therefore not accessible in the present calculations. The present results do demonstrate that the algebraic model is valid for this case, and thus lay a foundation for future studies that may resolve some of the discrepancies noted above.

9. Conclusions

I have used numerical solutions of the stability problem for a wide, double diffusive front in order to explore the potential for interleaving instability. Motion is assumed to take place on an \( f \)-plane. The background state is defined by uniform gradients of temperature, salinity and alongfront current. The state can include a nonzero cross-front buoyancy gradient, balanced by a baroclinic vertical shear. Perturbations are assumed to have the normal mode form, with vertical dependence expressed by a structure function to be computed. Parameter values are taken mainly from observations made on the lower flank of Meddy Sharon, but are varied to provide a more general view.

In the barotropic limit, the dominant mode describes stationary, oblique interleaving similar to that discussed by Stern (1967), Toole and Georgi (1981) and McDougall (1985) and others. While the plane of interleaving tilts in the alongfront direction, the motions are purely cross-front, as was shown by McDougall (1985).

With the introduction of baroclinicity, the oblique mode becomes oscillatory, i.e. it propagates in the alongfront direction. It also acquires vertical asymmetry, with maximum amplitude at the steering level. These modes appear in complex conjugate pairs consisting of oppositely propagating disturbances concentrated in the upper and lower halves of the domain. When baroclinicity is very weak, oblique modes retain the vertically quasi-periodic character of interleaving, modulated by an asymmetric envelope. The fact that only very weak baroclinicity is needed to break the symmetry of the interleaving mode

<table>
<thead>
<tr>
<th>fig.</th>
<th>( 10^3 s )</th>
<th>( h ) [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower flank observations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_p = 0 ) OB</td>
<td>3</td>
<td>5.7 ± 2.5</td>
</tr>
<tr>
<td>( s_p = 3.6 \times 10^{-3} ) CF</td>
<td>9</td>
<td>3.0</td>
</tr>
<tr>
<td>( Sc = 4 ) CF</td>
<td>13</td>
<td>5.4</td>
</tr>
<tr>
<td>( n_f = 2 ) CF</td>
<td>15</td>
<td>5.2</td>
</tr>
<tr>
<td>upper flank observations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_p = -1.0 \times 10^{-3} ) CF</td>
<td>19</td>
<td>-2.4 ± 1.3</td>
</tr>
</tbody>
</table>

Table 2: Summary of comparisons with Meddy Sharon observations. Parameter values from the lower and upper flanks of Meddy Sharon are from (Armi et al., 1989; Ruddick, 1992; May and Kelley, 2002). Computed modes are identified as cross-front (CF) or oblique (OB) and by figure number.
suggests that this effect may be common in the oceans. In fact, asymmetric interleaving is frequently observed (e.g. Grundlingh, 1985; Robertson et al., 1995). For the level of baroclinicity observed in meddy Sharon, the asymmetry is such that the quasiperiodic character is not evident. In all cases, motions remain primarily cross-front.

Baroclinicity also supports a class of interleaving modes whose growth rate is maximized at $l = 0$ and which I have called cross-front modes. These modes are stationary and vertically symmetric, with vertical wavelength smaller than that found in the barotropic case. The interleaving layers tilt only in the cross-front direction, but the motions within the layers are slanted toward the alongfront direction due to the Coriolis effect (figure 1). The growth rate and wavenumbers of the cross-front modes are well predicted by the “high shear” approximation of May and Kelley (1997). The growth rate is reduced if the saline diffusivity of salt fingering is allowed to decrease with increasing density ratio, in contrast to the result for the barotropic, nonrotating case (Walsh and Ruddick, 1995).

The cross-front interleaving modes have cross-front tilt very similar to the interleaving layers observed on the lower flank of Meddy Sharon, but their vertical wavelengths are smaller by an order of magnitude. This discrepancy is reduced if one assumes that the Schmidt number of salt fingering is greater than unity. The cross-front mode also differs from the observations in that the buoyancy perturbation is strongly dominated by its thermal component, a discrepancy that is resolved if $K_S$ is allowed to decrease with increasing density ratio, in contrast to the result for the barotropic, nonrotating case (Walsh and Ruddick, 1995).

The baroclinic front also supports the ageostrophic Eady mode. In all three mode classes (oblique, cross-front and Eady), the kinetic energy of interleaving is supplied mainly by the thermal buoyancy flux, while growth is opposed by some combination of dissipation and the saline buoyancy flux. Shear production is negligible for all cases studied here.

In double diffusive modes on weakly baroclinic fronts (e.g. Meddy Sharon), the buoyancy flux is driven by vertical advection of fluctuations in the salt flux due to salt fingering. Baroclinic buoyancy flux generation may be negative or positive, and becomes dominant in more strongly baroclinic fronts as the isopycnal slope approaches the isohaline slope.

When the flux ratio is allowed to vary with $R_\rho$, the cross-front mode appears to exhibit an ultraviolet catastrophe (growth rate increasing monotonically with vertical wavenumber) similar to that found by Walsh and Ruddick (2000) for the barotropic, non-rotating case. This behavior is clearly unrealistic and indicates a breakdown in the assumptions that underlie the mathematical model. For example, as a reviewer has pointed out, the parameterization of fluxes due to salt fingering is valid only on scales much larger than the size of an individual finger (10 cm). The fact that interleaving is observed on scales of meters or larger suggests that some other factor causes the model to break down at even larger scales. The mono-tonically increasing growth rate is reduced significantly when a constant diffusivity representing ambient turbulence is imposed. The fact that this damping is strong even when the ambient diffusivity is similar to molecular viscosity, i.e. $K_t = 10^{-6}$ m$^2$/s, suggests that molecular mixing processes could exert significant damping even in the absence of ambient turbulence. This turns out not to be the case; inclusion of molecular processes reduces the growth rate by only about 5%. This is because molecular damping is mainly the result of thermal diffusion, not viscosity, and molecular thermal diffusion is weak in water (i.e. the molecular Prandtl number is $\gg 1$). Another pos-sibility is that subharmonic instability causes interleaving layers to merge, forming the larger-scale layers seen in the observations.

In the diffusive convection regime, the physical processes driving the interleaving instability are dramatically different, but the stability characteristics are in many ways qualitatively similar to the salt fingering case. Most importantly, the symmetric interleaving mode is clearly evident and has maximum growth rate at $l = 0$.

In all cases examined here, the cross-front interleaving mode attains its maximum growth rate when the alongfront wavenumber is not significantly different from
zero, i.e. it is a true “cross-front” mode. As a result, the algebraic model used in previous studies, which assumes that $l = 0$, is shown to be valid. This result provides retroactive validation of previous work on baroclinic interleaving (e.g. Kuzmina and Rodionov, 1992; May and Kelley, 1997; Kuzmina and Zhurbas, 2000; May and Kelley, 2002; Kuzmina et al., 2005) and lays a foundation for future studies using the algebraic model. An issue of primary concern in this context is the UVC behavior of baroclinic interleaving when the flux ratio is allowed to vary realistically with $R_\rho$.

The boundary-trapped modes found here are potentially significant in the ocean. These modes depend strongly on the boundary conditions, however, which are not especially realistic in the present study. In auxiliary experiments I have replaced the conditions of constant temperature and salinity with zero-flux conditions, but the effect on the boundary-trapped modes was minor. A more realistic choice would be the reduced gravity constraints used to model the effect of a strongly stratified layer in studies of baroclinic instability in the mixed layer (e.g. Fukamachi et al., 1995; Beron-Vera and Ripa, 1997). Future studies must also deal with the effects of cross-front shear, i.e. $\partial V/\partial x \neq 0$. Interleaving motions are configured so as to advect this shear, generating new vertical shear and possibly a new mechanism for diapycnal mixing. Inclusion of cross-front shear in a model of cross-front modes is straightforward and has been done by May and Kelley (1997). In the more general model described here, $\partial V/\partial x \neq 0$ adds a new independent variable, $x$, which must be discretized, and therefore converts the problem to one of nonseperable stability analysis (e.g. Smyth and Peltier, 1991). The same is true of the aforementioned reduced gravity boundary conditions.

Finally, it is essential that this work be continued into the nonlinear regime. For cross-front modes (whose importance in the linear regime has been demonstrated here), this can be done by a straightforward extension of the one-dimensional model of Walsh and Ruddick (1998) and Mueller et al. (2006).

**Acknowledgements:** This work has benefited from discussions with Barry Ruddick and Bill Merryfield. The paper was clarified significantly thanks to the diligence of two anonymous reviewers. Funding was provided by the National Science Foundation under grant OCE0622922. I wish to thank the chorus of anonymous reviewers of a previous proposal who directed my attention to the problem considered here.

**Appendix A: The equivalent diffusivities.**

When the background stratification supports salt fingering, expressions for the effective diffusivities appearing in (14) follow directly from the expansions described in section 2c:

\[
\begin{align*}
K_{UU} &= K_t + ScK_S \\
K_{US} &= -(ScK_S)' R_\rho V_z B_{Sz} \\
K_{UT} &= -(ScK_S)' V_z B_{Sz} \\
K_{SS} &= K_t + K_S - R_\rho K_S' \\
K_{ST} &= -K_S' \\
K_{TS} &= -\gamma K_S + R_\rho(\gamma K_S)' \\
K_{TT} &= K_t + (\gamma K_S)'.
\end{align*}
\]

In the diffusive convection regime, the linearized equations of motions are identical to (14), but with effective diffusivities defined as follows:

\[
\begin{align*}
K_{UU} &= K_t + Pr K_T \\
K_{US} &= -(Pr K_T)' R_\rho V_z B_{Sz} \\
K_{UT} &= -(Pr K_T)' V_z B_{Sz}
\end{align*}
\]
\[ K_{SS} = K_t - (\gamma^{dc} K_T)'R_p^2 \]
\[ K_{ST} = -\gamma^{dc} K_T - (\gamma^{dc} K_T)'R_p \]
\[ K_{TS} = -K_T^{D'} R_p^2 \]
\[ K_{TT} = K_t + K_T + K_T^{D'} R_p \]  

In (24) and (25), \( K_S, \gamma, Sc, K_T, \gamma^{dc}, \) and \( Pr \) are functions of \( R_p \). Primes indicate the derivative \( d/dR_p \). All quantities are evaluated at the background value \( R_p = -B_T z / B_S z \).

### Appendix B: The Galerkin discretization

The \( z \)-dependence in (14) and (15) is discretized using a Fourier-Galerkin method. I define two sets of basis functions:

\[ F_n(z) = \cos \frac{n\pi \zeta}{H} ; \quad G_n(z) = \sin \frac{n\pi \zeta}{H} , \]  

where \( \zeta = z + H/2 \). In accordance with the boundary conditions at \( \zeta = 0 \) and \( \zeta = H \), the structure functions for vertical velocity and buoyancy are expanded as

\[ (\hat{w}, \hat{b}_i) = \sum_{n=1}^{N} (w_n, b_{in}) G_n(\zeta) , \]

while the horizontal velocity and pressure structure functions become

\[ (\hat{u}, \hat{v}, \hat{p}) = \sum_{n=0}^{N} (u_n, v_n, p_n) F_n(\zeta) . \]

The truncation parameter \( N \) determines the vertical resolution.

The expansions (27-28) are substituted into (14). The resulting five equations are then multiplied by \( F_m, F_m, G_m, G_m \), and \( G_m \), respectively, and the averaging operator

\[ \langle \rangle = \frac{1}{H} \int_0^H d\zeta \]  

is applied. The vertical dependence is thereby eliminated and the result is a set of linear, algebraic equations. Substitution of the Galerkin expansions into (15) yields an algebraic equation for the pressure coefficients \( p_n \). These are now substituted to obtain a closed set of equations for the coefficients \( u_n, v_n, w_n, b_{Sn} \) and \( b_{Tn} \). Concatenating these coefficients into a one-dimensional array allows one to write the system as a complex, general eigenvalue problem which can be solved using standard numerical methods.

### References


