Dynamic instability of stratified shear flow in the upper equatorial Pacific

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Abstract. The role of stratified shear instability in maintaining the deep cycle of turbulence immediately below the equatorial mixed layer is examined by means of linear stability analysis. The Taylor-Goldstein equation is solved numerically, using observed currents and stratification from the Tropical Instability Wave Experiment (TIWE). Multiple unstable modes are found, each associated with a local minimum of the gradient Richardson number. Wave radiation due to the unstable modes fluxes momentum downward through the Equatorial Undercurrent. The frequency, wavelength and propagation direction of the unstable modes are consistent with those estimated from observations of internal waves made during TIWE. These results, in combination with the previous observation that the internal waves are closely associated with deep cycle turbulence, lead us to conclude that shear instability is a dominant factor in the generation of the deep cycle.

1. Introduction

Shear instability has been suggested as a possible source of turbulent mixing in the stratified region above the Equatorial Undercurrent (EUC) core. A diurnal cycle in the turbulent kinetic energy dissipation rate, both in and below the surface mixed layer, was discovered during the Tropic Heat experiment [Moum and Caldwell, 1985; Gregg et al., 1985]. Intermittent bursts of intense turbulence were observed during nighttime in the stratified, low-gradient Richardson number (Ri) shear zone between the base of the mixed layer and the EUC [Peters et al., 1988; Moum et al., 1989]. The cause of the so-called "deep-cycle" turbulence was not known. It was speculated that high-frequency internal waves could be responsible for stimulating these bursts in the thermocline [Gregg et al., 1985; Moum et al., 1989]. In a further experiment, Tropic Heat 2, evidence of high-frequency, wavelike signals was found in the high shear zone (see Moum et al., 1992a) for a review) from both towed thermistor-chain observations [Moum et al., 1992b] and temperature time series from a mooring [McPhaden and Peters, 1992]. Microstructure measurements taken simultaneously with the towed thermistor observations showed the existence of turbulent overturns associated with packets of wavelike activity [Hebert et al., 1992]. The possible mechanisms for the observed wave signals were suggested as either downward traveling internal waves generated at the base of the mixed layer, the so-called "obstacle effect," or locally generated shear instability waves [e.g., Wijesekera and Dillon, 1991]. Here we will show that the unstable modes of the equatorial zonal current system are similar in spatial and temporal structure to the internal waves observed in association with the deep cycle. These results indicate that shear instability is the source of deep-cycle turbulence.

Our present knowledge of shear instability comes primarily from theoretical analyses and laboratory studies, which usually address simple flow geometries. In contrast, measurements from the ocean reveal complex profiles of current and stratification. While normal mode stability theory assures stability when Ri is everywhere greater than 1/4 [Miles, 1961; Howard, 1961], the onset of instability depends in a subtle way on the details of the velocity and density profiles [e.g., Howard and Maslowe, 1973]. Analytical solutions are only achievable for very few special profile functions [e.g., Drazin and Howard, 1966; Lott et al., 1992]. Hazel [1972] was the first to solve the problem numerically for a set of analytical functions. Subsequently, many authors have solved the stability problem with various analytical functions [e.g., Davis and Peltier, 1976; Smyth and Peltier, 1989; Winters and Riley, 1992; Sutherland and Peltier, 1992]. Sutherland [1996] studied the shear instability problem for the upper flank of the EUC using simple analytical functions to represent the background profiles.

As part of the Tropical Instability Wave Experiment (TIWE), an extensive data set including simultaneous measurements of microstructure, current, and internal waves was collected. The magnitude of Ri calculated
from the measurements frequently fell below 1/4 below the mixed layer [Lien et al., 1995]. This provides us with an excellent opportunity to study the shear instability problem with measured background conditions.

The objective of this paper is to investigate the characteristics of normal mode shear instability under realistic ocean conditions by solving the linear stability problem numerically, using background profiles derived from measurements during TIWE. We then address the degree to which these unstable modes can explain observed internal wave signals associated with the deep cycle of turbulence. We introduce the background condition for our analysis in section 2. In section 3, we provide a brief introduction to linear instability theory and present the eigenvalue problem we solve. The numerical method is presented in section 4. In section 5, the unstable modes for one single hour are examined in detail. This is followed in section 6 with a survey of the solutions over a 24-hour period. In section 7, we compare the characteristics of unstable modes from this analysis with related studies, including concurrent observations of internal waves, similar observations made during Tropic Heat 2, and other theoretical studies. A summary is presented in section 8.

2. Background Conditions

2.1. Data

During TIWE, R/V Wecoma occupied station at 0øN, 140øW for 21 days from November 4 to November 25, 1991. Intensive measurements of microstructure were made almost continuously day and night. Our microstructure profiler Chameleon [Moun et al., 1995] obtained measurements of conductivity, temperature, microstructure shear (from which turbulent kinetic energy dissipation rate $\varepsilon$ was computed), and pressure from near the surface down to about 200 m depth. Each individual microstructure profile took about 7–10 min to complete. A 150-KHz shipboard acoustic Doppler current profiler (ADCP) recorded current velocity, ensemble averaged over 5 min [Lien et al., 1994]. It used a 16-m pulse length and an 8-m bin width. The shallowest bin was centered at 20 m, and the deepest bin was at 524 m depth. The resulting ADCP velocities were smoothed volume averages of current velocity with 8-m vertical resolution. A conductivity-temperature-depth (CTD) cast was performed every other day down to 500 m depth.

Two towed thermistor chains (T chains) provided continuous time series of temperature profiles to about 50 m depth. The chains were deployed on the starboard side, one aft and one forward, with a horizontal spacing of 43 m. There were a total of 11 thermistors on each chain with vertical spacing of 3 m. The sampling interval was 10 Hz. Temperature time series from the T chains were averaged over 10 s. The two chains and the profiler were out of operation for 2 hours every second day while the ship was repositioned. The bow chain was at 10 m depth, the last one 40 m. On the stern–chain, the first thermistor was at 5 m, the last one at 35 m.

On November 9, 1991, an especially energetic wave event was observed from both T chains. Temperature time series from the thermistors at 35 m showed a remarkably well-defined wave structure that lasted for several hours (Figure 1a). Intense turbulence quantified by high values of $\varepsilon$ penetrated deeper after than before these waves appeared (Figure 1d). We thus choose the hourly averaged profiles of this 24-hour period for our stability analysis, with the intention of comparing the stability characteristics with observations.

2.2. Background Shear and Stratification

For the purpose of linear stability analysis, we define the "background flow" using hourly averaged profiles of currents and density measured by the ADCP and Chameleon, respectively. (This choice is discussed in detail in section 4.1.) In this section, we discuss the time-depth variations of several properties of the background flow that govern dynamic instability. These include zonal velocity $U(z)$, meridional velocity $V(z)$, the magnitude of the vertical shear $S = S^2 = U_z^2 + V_z^2$, and Brunt–Väisälä frequency (also called buoyancy fre-
Figure 2. Hourly averaged (a) zonal and (b) meridional velocity and (c) logarithm of squared shear $log_{10} S^2$ on the same day as in Figure 1. The solid line with circles represents the base of the surface mixed layer.

Figures 2 and 3 illustrate the upper ocean conditions that prevailed on November 9, 1991. In subsequent sections, we will use profiles of the hourly averaged data, shown in Figures 2 and 3 as input to the Taylor-Goldstein equation, and compute the stability characteristics of the water column throughout the 24-hour period. Above 50 m, the zonal flow was dominated by the westward flowing South Equatorial Current (SEC). We observed a well-developed EUC between about 100 m and 140 m, with maximum eastward speed near 1 m s$^{-1}$ (Figure 2a). The meridional velocity was much weaker, with maximum speed near 0.2 m s$^{-1}$, and was dominated by oscillatory motions (Figure 2b). The oscillation periods of the meridional velocity are at periods of 4, 8, and 20 days [Lien et al., 1995]. Strong shear was observed both above and below the EUC and in a third layer near 180 m depth (Figure 2c). Also, note the brief shear episode that began at hour 5 near 50 m depth. We will see that this episode led to intense shear instability. The strongest density stratification occurred near the surface at night in association with the diurnal cycle of surface forcing.

It is common practice to use the gradient Richardson number ($Ri = N^2/S^2$) to diagnose shear instability; that is, instability is expected if $Ri$ is smaller than $1/4$, based on the well-known theorem of Miles [1961]. We argue here that this approach is naive. The Richardson number only indicates the efficiency with which an instability may extract kinetic energy from the mean flow; one also needs a measure of kinetic energy available for extraction. Consider, for example, a flow in which the $N = 0$ and $S$ is nonzero but small. In this case, $Ri = 0$ and the flow may well be unstable, but growth rates of unstable modes may, nevertheless, be zero for all practical purposes. A better diagnostic of shear instability would take account of the absolute magnitude of the shear, as well as its magnitude relative to $N$. Such a diagnostic has been suggested by Kunze et al. [1990], who reanalyzed the theoretical results of Hazel [1972] to show that over a substantial range of $Ri$ growth rates of Kelvin-Helmholtz modes growing on a hyperbolic-tangent shear layer are nearly proportional to $S^{-2}$. In a separate publication (C. Sun et al., Parameterization of shear instability in the ocean thermocline, 2000), the best practical purposes.

Figure 3. Hourly averaged (a) logarithm of squared buoyancy frequency $log_{10} N^2$ (b) inverse-gradient Richardson number $Ri^{-1}$, and (c) reduced shear $S - 2N$ on the same day as in Figure 1. The solid line with circles represents the base of the surface mixed layer.
manuscript in preparation, 1998)(hereinafter referred to as Sun et al., manuscript in preparation, 1998), we show that this proportionality is also approximately valid for the much more complex flow geometries found in the ocean. We refer to the quantity \( S - 2N \) as the reduced shear and make frequent use of it in interpreting our results.

Subcritical values of \( Ri \) (i.e., \( Ri < 1/4 \)) were observed at approximately 30–90 m depth throughout the 24-hour period of study (Figure 3b) and intermittently at the deep shear layer located near 180 m depth. These coincided (by definition) with positive values of reduced shear \( S - 2N \) (Figure 3c). The smallest values of \( Ri \) occurred at hour 9 near 35 m depth. However, values of the reduced shear were not especially large at that point. The largest value of the reduced shear occurred at hour 5 near 50 m depth. We will see that this is the location where the water column was most unstable.

3. Eigenvalue Problem

For the present computations, we employ the classical method of linear stability analysis based on normal modes. Normal mode analysis is the basis of most of the common theoretical results which are used in the interpretation of ocean turbulence, for example, the Miles-Howard stability criterion [Miles, 1961; Howard, 1961]. The general applicability of the normal mode approach has recently been called into question in light of its failure to accurately predict the transition to turbulence in certain non-inflectional shear flows, for example, plane Couette and Poiseuille flows. In these cases, transition has been observed at points in parameter space lying well outside the stability boundary predicted by normal mode theory and has been attributed to the non-modal growth of superpositions of neutral modes from the continuous spectrum [e.g., Butler and Farrell, 1992; Trefethen et al., 1993]. In contrast, transition phenomena in inflectional background flows are explained quite adequately in terms of normal modes [e.g., Thorpe, 1971; Scotti and Corcos, 1972]. This does not prove that non-modal phenomena are not important in such flows, but it does inspire confidence that normal mode analysis captures the dominant physical mechanisms of transition.

We consider the three-dimensional (3-D) Taylor-Goldstein equation [e.g., Boyd et al., 1993], where the background flow has both zonal and meridional velocity components. The details of derivation of the Taylor-Goldstein equation are presented in appendix A. We assume a normal mode solution,

\[ [u', v', w', p', \rho'] = \text{Re} \{ [\hat{u}(z), \hat{v}(z), \hat{w}(z), \hat{p}(z)] e^{i(kx + ly - \omega t)} \} \quad (1) \]

where primed variables represent perturbations from the background state as defined in section 2. \( \mathbf{K} = (k, l) \)

and meridional wavenumbers, respectively. The amplitude of the horizontal wave vector is \( \kappa = \sqrt{k^2 + l^2} \); \( \omega = \omega_r + i\omega_i \) is the complex frequency. The variable \( \omega_r \)

is the fixed frequency of the normal mode in a motionless frame; \( \omega_i \) is the growth (or decay) rate.

For an inviscid, incompressible, stratified, Boussinesq shear flow, the 3-D stability equation is

\[ \hat{w}_{xx} + \left( \frac{N^2}{(U - c)^2} \right) \hat{w} = 0. \quad (2) \]

where \( U(z) \) and \( V(z) \) represent mean velocity components in the zonal and meridional directions, respectively, i.e. the mean flow is \( U(z) = (U(z), V(z), 0) \). \( \hat{w} \)

is the component of the mean flow parallel to the horizontal wave vector: \( \hat{w} = \hat{w}(z) \). \( \theta = \tan^{-1}(l/k) \) is the direction of the wave vector. The complex phase velocity is \( c = c_r + ic_i = \omega / \kappa \), and \( \omega_i \) represents the growth (decay) rate of an unstable mode if \( c_i > 0 \) (< 0). For future reference, a critical level for a given normal mode is a depth where the current velocity \( \hat{U}(z) \) matches the phase speed \( c_r \). It is the nearest point on the real \( z \) axis to the singularity at \( \hat{U} = c \).

At the ocean surface, we require the vertical velocity of the perturbation to be zero. A radiation boundary condition is imposed at the lower boundary (located at 200 m depth for the present study), so that waves generated at the shear layer can propagate downward into the thermocline (appendix B). The Taylor-Goldstein equation and the boundary conditions constitute an eigenvalue problem which implicitly defines the dispersion relation \( c = c(K) \) or, equivalently, \( \omega = \omega(k, l) \).

4. Numerical Methods

We apply a shooting method [Press et al., 1992] to solve the eigenvalue problem, similar to that used by Hazel [1972]. We modified it to account for the complexity of the background profiles used here. This shooting method integrates (2) from one boundary to the other, using a fifth-order adaptive step-size Runge-Kutta scheme. For given values of \( \kappa \) and \( \theta \), as well as an initial guess for \( c \), it tries to match the boundary condition at the other boundary by adjusting the value of \( c \). It thus defines a matching function with \( c \) being the independent variable. A globally convergent Newton’s method is utilized to locate the complex eigenvalue \( c \).

4.1. Basic States for the Linear Stability Analyses

Stability analysis via the Taylor-Goldstein equation requires that we specify a steady, parallel shear flow to represent the basic state. This is problematical, since such a flow is an idealization that does not occur in nature. In our case, the observed flow consists of a complicated superposition of horizontal flow plus internal waves and turbulence. Our goal is to isolate the verti-
cally sheared, horizontal flow associated with the large-scale equatorial current system in order to see whether its unstable modes can shed light on the physics of the observed waves and turbulence. We must therefore apply a decomposition operation to the measured currents and density. The choice we make is a 1-hour running time average (combined with vertical smoothing to be described below). Smyth and Peltier [1994] have shown that, on a time-varying background flow, the growth of unstable modes over a finite interval \( \Delta t \) can be predicted via stability analysis of the background flow averaged over \( \Delta t \). The results become exact for growth rates \( \gg 1/\Delta t \). Lien et al. [1996] showed that wave signals observed from a moored thermistor chain had frequencies greater than 1 cph at our location and about the same time. The 1-hour averaging interval is therefore a reasonable choice for the definition of the basic state. Insofar as the temporal and spatial structures of the resulting normal mode instabilities compare favorably with observed fluctuations, we may reasonably conclude that the latter are forced by the former. As a result, we can understand the observations in terms of a substantially reduced subset of the physical phenomena that are known to occur in this oceanic regime.

To obtain a continuous distribution of velocity and density as functions of depth, we employ piecewise polynomial functions to interpolate between the discrete values provided by the data. Since second-order derivatives of velocity \( U_{zz} \) and \( V_{zz} \) are needed, we choose (natural) cubic splines for the interpolation. As we have mentioned in section 2, the velocity data of the shipboard ADCP are vertical profiles recorded at intervals of 8 m from 20 m to 500 m depth. From the surface to 20 m depth, the velocity is linearly extrapolated. The velocity thus obtained was compared with measurements by current meters at depths of 3 m and 10 m from a nearby National Oceanic and Atmospheric Administration Pacific Marine Environmental Laboratory (PMEL) mooring. Discrepancies between extrapolated currents and PMEL currents are not significantly larger than discrepancies between our measured currents at deeper depths and the corresponding PMEL currents, indicating that the extrapolation is not a significant source of error. Below 20 m, the velocity is interpolated with splines. The resulting second-order derivatives of velocity \( U_{zz} \) and \( V_{zz} \) are piecewise linear functions of depth over every 8-m interval.

To match the inherent filtering of current velocity by the ADCP with a 16-m pulse length [Lien et al., 1994], hourly averaged density profiles are smoothed by a 16-m triangular filter. The filtered density profiles are sampled at 4-m intervals, then interpolated using (natural) cubic splines. (This slight mismatch in processing of currents and density is necessary because of spurious convective overturns resulting from cubic spline interpolation for density profiles in 8-m vertical intervals.)

To properly manage the discontinuities in the derivatives of \( U_{zz} \) and \( V_{zz} \), the integration of (2) is done in steps from one data point to the next, so that the evaluation of the derivatives of \( U_{zz} \) and \( V_{zz} \) across data points is avoided. Other types of cubic splines have been tested (e.g., a scheme suggested by Akima [1970]) and resulted in less than 10% differences in computed growth rates. The computation converges faster than other cubic spline schemes for natural splines, presumably because of its smooth second-order derivatives \( U_{zz} \) and \( V_{zz} \).

The lower boundary is set at 200 m depth in this study, which is the deepest depth of the microstructure profiles. We have also calculated the stability solution with the lower boundary at 500 m depth, using information from CTD casts. Results are similar to the case when the boundary is at 200 m depth; the growth rates of the fastest growing modes differ by less than 1%.

### 4.2. Range of Wavenumbers

Our objective is to obtain the eigenvalue \( c \) for given wavenumbers. For each choice of \( \kappa \) and \( \theta \), we solve the equivalent two-dimensional problem with the profiles of mean horizontal velocity vector projected on a vertical plane in the direction \( \theta \), that is, \( \bar{U}(z) \). We first need to decide the range of directions to be investigated. As shown in appendix B, the Miles-Howard theorem is valid for the present case. As a result, if \( \bar{R} \) is greater than 0.25 at every depth, the flow is stable, and no further calculations are needed. If the projected mean flow in a particular direction \( \theta \) has \( \bar{R} \) \( (\bar{R} = N^2/\bar{U}^2) \) greater than 0.25 at every depth, the flow is stable to small perturbations whose wave vector is in that direction. We therefore perform the calculation only for \( \theta \) such that \( \bar{R} \theta < 1/4 \) at some \( z \). Also, by symmetry, only half the wavenumber domain needs to be considered; we choose the range of \( \theta \) to be \([-90^\circ, 90^\circ]\).

The range for wavenumber magnitude \( \kappa \) is not as clearly defined as that for \( \theta \). For hyperbolic tangent profiles of velocity and density, the wavelength of the unstable waves is proportional to the shear-layer depth, that is, the depth over which the shear is large. The wavelength of the fastest growing instability ranges from 4 to 8 times the shear-layer depth [e.g., Drazin, 1958; Miles and Howard, 1964; Davis and Peltier, 1976], depending on the details of the profiles and the boundary conditions. For the complicated profiles considered in this paper, there is no well-defined "shear layer," but our background currents tend to vary on scales of a few tens of meters. Accordingly, we restrict the stability analysis to values of the wavenumber \( \kappa \) ranging from 5 \( \times \) \( 10^{-4} \) rad m\(^{-1}\) to 10\(^{-1}\) rad m\(^{-1}\) at intervals of 5 \( \times \) \( 10^{-4} \) rad m\(^{-1}\). The corresponding wavelengths range from 63 m to 12,570 m. This includes the range of wavelengths estimated by Moum et al. [1992b] from towed thermistor-chain data, by McPhaden and Peters [1992], and by M.D. Levine and J.N. Moum (Detailed observations of an equatorial internal wave packet, manuscript in preparation, 1998)(hereinafter referred to as Levine...
and Moum, manuscript in preparation, 1998) from data obtained in this experiment.

### 4.3. Initial Guesses for the Eigenvalue

The shooting method requires an initial guess for the complex eigenvalue \( c \). We look only for the eigenvalues with a positive \( c_i \), which represents growth with time. The bounds on the complex phase speed \( c \) depend on the range of velocity and are given by Howard's semicircle theorem (appendix B). For the nonparallel background flows considered here, the range of velocity is, in turn, a function of \( \theta \). To find all possible unstable modes, we provide a set of initial guesses (typically 10) chosen to lie inside Howard's semicircle.

### 5. Detailed Study of Unstable Modes from a Single Hour

We first investigate the unstable modes for one hour in detail, then we summarize the general properties of unstable modes for every hour of November 9, 1991 (calendar day 313). The growth rate is a maximum at Hour 5 (2000 local time). Owing to incomplete data at hour 5, we choose hour 6 (the hour with the second largest growth rate) as a focus for detailed study in the remainder of this section.

#### 5.1. Mean Profiles During Hour 6

During hour 6, the zonal flow above 50 m was dominated by the westward flowing South Equatorial Current (Figure 4a). The core of the eastward flowing EUC was located between about 100 and 120 m, with a maximum speed near 0.8 m s\(^{-1}\). Meridional flow was southward above 150 m and exhibited oscillatory structure in the vertical (Figure 4b). The largest shears occurred near 50 m and 80 m. Though the total shear was dominated by its zonal component, meridional shear contributed to the shear maxima at these two depths. \( N^2 \) was largest around 140 m (Figure 4c). The high shear and relatively weak stratification resulted in a maximum reduced shear \( S - 2N \) near 50 m (Figures 4d and 4e). The stratification near 80 m was strong enough to cause \( Ri \) to be greater than the critical value \( 1/4 \) there. Three regions had positive reduced shear (or \( Ri < 1/4 \)): the surface mixed layer (from the surface to about 16 m), deep-cycle region (28-57 m), and a thin layer below both the EUC and the main pycnocline (173-181 m) (Figure 4f).

In the surface mixed layer, \( Ri \) dropped to nearly zero, and reduced shear was large, suggesting strong instability. However, instability can be damped by boundary proximity effects [Hazel, 1972]. That appears to be the case here; we will see that the deep cycle layer and the deep shear layer at 180 m are the main sites of instability.

![Figure 4](image-url)
5.2. The Dispersion Relation

Typically, the dispersion relation $\omega = \omega(k, l)$ defined in section 3 is not single valued. Instead, it exhibits several overlapping surfaces upon which $\omega$ varies continuously. On each surface, the growth rate, $\omega_0 = \kappa c_t$, varies significantly, while the phase speed $c_p = \omega/c_0$ is relatively constant. What this means physically is that each surface represents a continuum of unstable modes whose critical levels are close to a given depth (usually a depth at which reduced shear is large and $Ri$ is small). We refer to each of these continua as a "family" of unstable modes. Each family has a "fastest growing mode" (FGM), the mode whose wave vector $(k, l)$ is such that the transfer of energy from the mean flow is optimized and thus the growth rate is a maximum. This is the mode that is expected to dominate in the long-time limit. In reality, disturbances grow over finite times, so that modes other than the FGM will also have significant amplitude. As a result, each mode family is expected to generate a wave packet. The horizontal scale of the wave packet is inversely proportional to the width of the growth rate maximum on the $(k, l)$ plane and grows with time, so that the idealization of a monochromatic wave train is realized in the limit $t \to \infty$.

Three families of unstable modes can be identified for hour 6 according to their critical levels (Figure 5). We denote the mode family with the shallowest critical level, in the westward flowing SEC at 35 m depth, as mode 1. By definition, mode 1 thus has westward phase velocity. Mode 2 refers to the mode family with a critical level at 47 m, which is at the boundary of the SEC and EUC. Therefore mode 2 is almost stationary. Its dispersion relation has a bimodal structure, with comparable growth rates for the two closely connected peaks. Mode 4 has a deep critical level, located at the lower flank of the EUC near 176 m. Mode 4 thus has eastward phase velocity. It has two separate peaks, and their growth rates are quite different. (As will be seen in Section 6, mode 3 is important at times other than hour 6, and is located between modes 2 and 4 vertically in the water column.) For simplicity, hereafter we refer to the FGM of each mode family as the mode; for example, the FGM of mode 1 will be referred to as "mode 1."

The growth rates and phase velocities of these unstable modes are very different. Mode 1 has the smallest growth rate $4 \times 10^{-4}$ s$^{-1}$ and propagates to WSW with a speed of 0.3 m s$^{-1}$. Mode 2 has the largest growth rate, $2.1 \times 10^{-3}$ s$^{-1}$, with a phase speed very close to zero. Mode 4 propagates to ENE with a speed of 0.2 m s$^{-1}$. Its growth rate was $6.4 \times 10^{-4}$ s$^{-1}$. The wavelengths of these unstable modes range from about 80 m to 450 m (Figure 6). The characteristics of these three unstable modes of the mean flow observed during hour 6 are summarized in Table 1.

Figure 5. Growth rate and critical level versus zonal and meridional wavenumbers $k$ and $l$. The height represents the growth rate. Shading represents the critical level depth for each mode. The numbers 1, 2, and 4 refer to modes 1, 2, and 4. Each mode is characterized by a different phase speed according to the depth of its critical level.

Figure 6. Growth rate $(\omega_i \times 10^{-4}$ s$^{-1}$) versus zonal and meridional wavenumbers, $k$ and $l$, respectively, for hour 6. The contour interval is $10^{-4}$ s$^{-1}$. Half circles represent constant wavelengths.
Table 1. Characteristics of Unstable Modes of Hour 6

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\lambda$, m</th>
<th>$\theta$, deg</th>
<th>$\omega_t$, s$^{-1}$</th>
<th>$T_e$, min</th>
<th>$c_r$, m s$^{-1}$</th>
<th>$\omega_r$, cph</th>
<th>$z_c$, m</th>
<th>$z_R$, m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>300</td>
<td>195</td>
<td>$4.0 \times 10^{-4}$</td>
<td>41.7</td>
<td>0.30</td>
<td>3.6</td>
<td>35</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>188</td>
<td>335</td>
<td>$1.1 \times 10^{-3}$</td>
<td>15.3</td>
<td>0.00</td>
<td>0.0</td>
<td>47</td>
<td>46</td>
</tr>
<tr>
<td>4</td>
<td>106</td>
<td>10</td>
<td>$6.4 \times 10^{-4}$</td>
<td>26.0</td>
<td>0.20</td>
<td>6.8</td>
<td>176</td>
<td>177</td>
</tr>
</tbody>
</table>

Variable $\lambda$ is the wavelength of the unstable mode, $\theta$ is the direction of its propagation, and $\omega_t$ is the growth rate. $T_e$ is the corresponding e-folding time. Variable $c_r$ is the real part of the complex phase speed, $\omega_r$ is the frequency of unstable mode relative to the ground, $z_c$ is the depth of critical level, and $z_R$ is the depth of local $R_i$ minimum nearest to the critical level of each unstable mode.

5.3. Vertical Structure of Unstable Modes

Because (2) and its boundary conditions are linear and homogeneous, the absolute value of the amplitudes of the eigenfunctions cannot be determined. We normalize the solutions by scaling the maximum amplitude of the vertical velocity eigenfunction to unity. Velocity structures for modes 1 and 2 have significant amplitude throughout the 200-m domain (Figure 7a). The amplitude of $\dot{w}$ for mode 2 goes to zero near the lower boundary, while the amplitudes of $\dot{w}$ for modes 1 and 4 remain substantial (Figures 7b and 7c), suggesting significant radiation of energy through the lower boundary. Interestingly, the largest amplitude of vertical velocity of mode 1 occurs at about 100 to 120 m, coincident with the core of the EUC. The vertical velocity of mode 4 only has significant amplitude below about 70 m but is large at the lower boundary. Profiles of vertical velocity indicate a typical Kelvin-Helmholtz structure: a minimum at the critical level and maxima at either side [De Bass and Driedonks, 1985]. The phase of $\dot{w}$ changes across the critical level but is relatively constant above and below the critical level.

To better understand the physics of unstable modes, we examine the depth-dependent kinetic energy equation

$$\frac{1}{2}(u'^2 + v'^2 + w'^2) = -u'u'U_z - v'v'V_z - gp'w' - (p'w')_z$$

where the overbar denotes average over one horizontal wavelength. The derivation of (3) is given in appendix A. The terms on the right-hand side represent shear production, work done by gravity, and the convergence of the vertical energy flux, respectively. Both shear production and gravitational work are strongly concentrated at the critical level (Figure 7, second and third columns). The vertical flux of perturbation kinetic energy $p'w'$ is positive above the critical level and is negative below it; that is, perturbation kinetic energy is transported vertically away from the critical layer. For modes 1 and 2, kinetic energy is extracted from the shear zone between the surface and the EUC and redistributed throughout the vertical domain. For mode 4, the energy flux $p'w'$ has significant magnitude at the

![Figure 7](image-url)
lower boundary. This indicates that this mode is capable of transporting energy vertically into deeper regions. The buoyancy flux due to the disturbance $\rho'w'$ is positive throughout the domain. This is anticipated because in stable stratification a part of the kinetic energy extracted from the background shear is used to increase the potential energy of the flow.

We now seek to identify the particular characteristics of the background profiles at hour 6 (see Figure 4) that govern the growth rates and phase velocities of the unstable modes. To do this, we plot the growth rates of all of the unstable modes that our shooting code has identified at hour 6 (Figure 8). The vertical location of each point corresponds to the critical level of a particular mode; the horizontal coordinate is its growth rate. The modes are superimposed on profiles of various background quantities. Critical levels of unstable modes tend to cluster in thin layers surrounding the maxima of the reduced shear (which tend to coincide with minima of $Ri$). It is clear from these results that the reduced shear and the Richardson number are the best indicators of instability.

6. Variability of Shear Instability Over a 24-hour Period

We now examine the stability characteristics of the upper ocean during a 24-hour period, which includes the single hour considered in detail already (Figure 9). Since there were no microstructure measurements during hour 5, we interpolated the background density profile from the two neighboring hours (velocity data were available from the ADCP). This case produced the most unstable mode for the 24-hour period. To be sure that the averaging of the density profiles had not introduced artificial effects in the stability analysis, we repeated the calculations using the density profile of hour 4 alone then repeated the calculations using the hour 6 profile. The fastest growing modes calculated using the density profiles of hours 4 and 6 differ by only 2–4% from that of calculations using the interpolated density profile. This assures us that hour-to-hour changes in stability properties are governed mostly by changes in shear, so that approximation of the missing density profile does not give misleading results.
We organize our results by defining four regimes of instability, based on the zonal current structure (see Figure 9c).

1. Nearest the surface, the westward flowing SEC was unstable throughout hours 6–11 then became stable as local Ri increased above 1/4. We refer to this instability of the near-surface region (i.e. to unstable modes having critical levels in this layer) as "mode 1" (cf. Section 5.2). Mode 1, by definition, has zonal phase velocity directed to the west (Figure 10b). In section 7 we will show that this mode explains concurrent observations of wavelike disturbances.

2. Our second regime of instability lies on the boundary between the SEC and the eastward flowing EUC, where the zonal velocity is near zero. We refer to instabilities with critical levels in this region as "mode 2"; more specifically, the critical level of a mode 2 instability lies between the $U = -10$ cm s$^{-1}$ and $U = 10$ cm s$^{-1}$ isotachs. This regime was continuously unstable from hour 5 to hour 9 then became unstable again after hour 12 (Figure 10c). The most energetic instabilities found in these calculations fall into the mode 2 category. The largest growth rate was obtained from background profile at hour 5, $2.1 \times 10^{-3}$ s$^{-1}$ (corresponding to an e-folding time of less than 8 min). There were also strong instabilities in hours 6 and 7, with growth rates about $1.1 \times 10^{-3}$ s$^{-1}$ (corresponding to an e-folding time of about 15 min). These episodes of intense instability occurred during the evening (hours 5, 6 and 7 correspond to local time 2000, 2100, and 2200, respectively).

3. Our third class of modes had critical levels located on the upper flank of the EUC and thus had eastward phase velocity (Figure 10d). This mode was unstable only intermittently and exhibited very small growth rates during those times.

4. Our final class of instabilities, mode 4, was focused in the thin shear layer at the bottom of the EUC near 180 m depth. Like mode 3, this mode was weak and only intermittently unstable (Figure 10e).

Figure 9. (a) Hourly averaged buoyancy flux $J_0$ on November 9, 1991. (b) Growth rates of the unstable modes for each hour. Squares represent the growth rate of mode 1, triangles represent mode 2, circles represent mode 3, and diamonds represent mode 4. (c) Image and contours of the zonal velocity. The squares, triangles, circles, and diamonds now indicate the location of critical level of each mode. (d) Image of the reduced shear $S - 2N$. The squares, triangles, circles, and diamonds have the same meaning as in previous frames.

Figure 10. (a) Hourly averaged buoyancy flux $J_0$ on November 9, 1991. The phase velocity is shown in (b) mode 1, (c) mode 2, (d) mode 3, and (e) mode 4. The arrows indicate the phase velocities of the unstable modes, originating at the hour of the calculation. A 10 cm s$^{-1}$ scale is shown in Figure 10e. The direction of the arrow represents the direction of propagation (right is east, and up is north). (f) The frequency of unstable modes seen by a stationary observer. (g) The wavelengths of all the unstable modes. The squares, triangles, circles and diamonds in Figures 10f and 10g have the same meaning as in Figure 9.
7. Comparison With Related Studies

7.1. Concurrent Observations of Internal Waves

An especially energetic wave event was recorded by both thermistor chains during hours 7, 8, and 9 (Figure 1). Levine and Moum (manuscript in preparation, 1998) studied this wave packet in detail and estimated its frequency to be about 5 cph in the reference frame of the moving ship. A wave packet was simultaneously observed by a nearby mooring [Lien et al., 1996]. The estimated frequency from the mooring was 2–3 cph. Assuming the waves observed were instability waves triggered by random perturbations, we compare the frequencies of our unstable modes as seen by a stationary observer and as seen from the ship’s reference frame with the observations.

Data needed for this comparison are summarized in Table 2. We will consider modes 1 and 2 in turn. Mode 1 at hour 9 has frequency 3.3 cph in the reference frame of the Earth, which becomes 4.2 cph in the reference frame of the moving ship. These frequencies compare well with the observations. However, T chain records show similar wavelike activity during the previous hour. The shallow mode at hour 8 has frequency 3.2 cph in the Earth frame, but in the ship’s frame, this becomes 2.4 cph, a poor fit to the observation. During hours 6 and 7, however, the calculated frequencies of mode 1 agree well with the observations. The frequencies in the Earth’s frame are 3.6 and 3.4 cph for hours 6 and 7, respectively. They become 6.8 and 5.9 cph in the ship’s frame. Because these instabilities grow with e-folding times of the order of an hour, it is plausible that there will be a time lag of a few hours between the onset of instability and the observation of large-amplitude waves. On the whole, there appears to be a significant correspondence between mode 1 and the wavelike oscillations observed at the same time.

The wavelength of mode 1 is about 300 m. This is consistent with the estimates by Levine and Moum (manuscript in preparation, 1998) and Lien et al. [1996]. Levine and Moum’s estimate of the wavelength is around 400 m. Lien et al. estimated a wavelength of 200–350 m from the mooring measurement, assuming a westward propagating speed of 0.3 m s⁻¹ (from estimate by Levine and Moum). The wavelength of mode 1 also agrees with observations of similar events during Tropic Heat 2, as discussed later.

The observations could also have been associated with the mode 2 instability. The latter was focused in a depth range not much deeper than mode 1 and had larger growth rates. However, the frequencies do not compare as well with the observations. At hour 5, mode 2 had a very large growth rate, and its frequency in the ship’s frame was 4.7 cph, very close to that observed a few hours later in the T chain signals. This correspondence suggests that the observed waves could have been a remnant of the mode 2 instability at hour 5. However, the frequency of that mode in the Earth’s frame was much smaller than that observed at the mooring. In subsequent hours, mode 2 exhibits frequencies in the reference frames of both motionless Earth and moving ship that compare poorly with the observations. These results suggest that the observed waves are, indeed, remnants of the mode 1 instability. It could be that the relatively small growth rates assigned to mode 1 are inaccurate, possibly because near-surface shear was underestimated by the ADCP, as suggested by Mack and Hebert [1997].

7.2. Internal Waves Observed During Tropic Heat 2

We have shown that shear instability is capable of generating coherent structure over large vertical extent, far beyond the zone of small Ri. This kind of

<table>
<thead>
<tr>
<th>Hour</th>
<th>Mode</th>
<th>( \kappa ), rad m⁻¹</th>
<th>( \theta ), deg</th>
<th>( c_p ), m s⁻¹</th>
<th>( \omega_i ), s⁻¹</th>
<th>( \omega_r ), cph</th>
<th>( \omega_{\text{ship}} ), cph</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
<td>0.0325</td>
<td>325</td>
<td>0.042</td>
<td>2.1x10⁻³</td>
<td>0.8</td>
<td>4.7</td>
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<tr>
<td>6</td>
<td>1</td>
<td>0.0210</td>
<td>195</td>
<td>0.096</td>
<td>4.0x10⁻⁴</td>
<td>3.6</td>
<td>6.8</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0335</td>
<td>155</td>
<td>0.092</td>
<td>1.1x10⁻³</td>
<td>0.9</td>
<td>2.2</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.0195</td>
<td>150</td>
<td>0.030</td>
<td>2.3x10⁻⁴</td>
<td>3.4</td>
<td>5.9</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0315</td>
<td>155</td>
<td>0.054</td>
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<td>1.0</td>
<td>1.1</td>
</tr>
<tr>
<td>8</td>
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<td>0.027</td>
<td>3.1x10⁻⁴</td>
<td>0.4</td>
<td>3.2</td>
</tr>
</tbody>
</table>

Variable \( \kappa \) is the magnitude of the horizontal wavenumber of the unstable mode, \( \theta \) is the direction of the unstable mode, and \( c_p \) is the phase speed. Variable \( \omega_i \) is the growth rate, \( \omega_r \) is the frequency of unstable modes relative to the ground, and \( \omega_{\text{ship}} \) is the frequency of the unstable modes that would be observed from a ship moving with velocity \((0.13, 0.55)\) m s⁻¹, the velocity of the R/V Wecoma during this period.
vertical structure is consistent with the observations of wave events during Tropic Heat 2. Mourn et al. [1992b] showed that the vertical scale of an energetic wave event observed during Tropic Heat 2 was greater than 120 m, much larger than the vertical extent of the low Richardson-number region. The wavelengths of unstable modes from our results are consistent with the observations of similar events during Tropic Heat 2. The wavelength of the wave event studied by Hebert et al. [1992] was about 100 m. From 4 days of observations of high-frequency internal waves in the same experiment, Mourn et al. [1992b] estimated that the scale of vertical coherence of this band of waves exceeded 120 m. They found that the spectrum of vertical isotherm displacements was dominated by a narrow wavenumber band of internal waves with wavelengths between about 150 to 250 m. McPhaden and Peters [1992] estimated that the zonal wavelength of high-frequency waves observed from a mooring was about 100–300 m. These estimated wavelengths are consistent with our results.

7.3. Theoretical Studies

Our study complements the work of Skillingstad and Denbo [1994], who studied the generation and evolution of the internal waves using a two-dimensional nonlinear model. One of the simulations they performed applied the initial state from the measured velocity and density profiles on the equator near 134°W reported by Hebert et al. [1991]. They suggested that shear instability in the mixed layer was responsible for the generation of internal gravity waves. The wavelength of the instability wave was about 250 m, with phase speed about -0.03 m s⁻¹. These values of wavelength and phase speed are similar to those of mode 2 in our study. Our results therefore indicate that the instabilities that Skillingstad and Denbo [1994] attributed to the surface mixed layer could also be generated in the stratified layer just below the mixed layer.

Two more recent theoretical analyses have focused on shear instability at the same site. Modeling the upper flank of the EUC as a hyperbolic tangent velocity profile, Sutherland [1996] solved the stability problem and showed that shear instability may generate downward propagating internal gravity waves, with a wavelength of about 100 m and phase speed 0.5 m s⁻¹ eastward. Mack and Hebert [1997] solved the same problem using more complicated analytical functions to approximate the background conditions during Tropic Heat 2. Because the measured Ri did not fall below 1/4, they enhanced zonal shear in the upper 40 m to force Ri to be smaller, arguing that near-surface shears were underestimated by available measurement techniques. They found that the e-folding time for the most unstable mode for some cases was less than 10 min. The wavelength ranged from about 100 to 300 m. The phase speed was always westward in the range of -0.8 to -0.1 m s⁻¹.

Our analyses are reasonably consistent with these two studies. Like Mack and Hebert [1997], we find that a strongly sheared westward current near the surface leads to westward propagating unstable modes (mode 1). The differences in our case are that (1) the strong near-surface shears were not hypothetical but observed by the combination of shipboard ADCP and the PMEL mooring and (2) the dominant instabilities are focused at the boundary of SEC and EUC (about 50 m depth) rather than near the surface. Like Sutherland [1996], we have examined a situation in which sub-critical Richardson numbers exist in the upper flank of the EUC. We find eastward propagating unstable modes that radiate energy from the upper EUC into the deep ocean. To extend the results of Sutherland [1996], we have employed realistic velocity and density profiles, both above and below the EUC core, so that the complex wave propagation characteristics of the lower EUC and upper thermocline are represented in detail.

8. Summary

During TIWE, we made simultaneous measurements of velocity and density profiles in the mixed layer and upper thermocline at 0°, 140°W. Here we have solved the three-dimensional Taylor-Goldstein equation numerically using hourly averaged density and velocity profiles measured on November 9, 1991. We have shown that the sheared zone between the base of the surface mixed layer and the core of the EUC (the deep-cycle layer) can exhibit dynamic instability. A thin shear layer at the lower edge of the EUC was also unstable.

The largest growth rate found in this study is 2.1 x 10⁻³ s⁻¹, which corresponds to an e-folding time of less than 8 min. The frequencies of these unstable modes are of the order of a few cycles per hour (Figure 10f). The propagation directions of these unstable modes are mostly zonal (Figures 10b, 10c, 10d, and 10e), with phase speeds between -0.3 and 0.4 m s⁻¹. The wavelengths of unstable modes range from 100 to 400 m (Figure 10g). These wavelengths are a few times the depth of the shear zone, as would be expected on the basis of analysis of simpler profiles (see section 4.2).

The temporal and spatial characteristics of the unstable modes of the hourly averaged flow are consistent with the corresponding parameters of internal waves observed in the same regime. The connection between those internal wave signals and deep-cycle turbulence has been established elsewhere [McPhaden and Peters, 1992; Mourn et al., 1992a, b]. Our results are therefore consistent with the scenario in which shear instability is responsible for the maintenance of the deep-cycle turbulence, as speculated by Gregg et al. [1985], Peters et al. [1989], Mourn et al. [1992a, b] and McPhaden and Peters [1992]. Small disturbances extract energy from the mean flow near the critical level through the interaction of Reynolds stress and mean shear. The kinetic energy is redistributed over large vertical extent, so that distur-
bances have significant amplitude far beyond the region of subcritical Richardson number. Through the action of these unstable modes, westward momentum from the wind-driven near-surface current (the SEC) is carried downward and distributed throughout the vertical extent of the EUC and beyond. These unstable modes and the associated waves and turbulence may therefore play a significant role in the momentum balance that maintains the equatorial zonal current system.

Appendix A: Polarization Relations and the 3-D Taylor-Goldstein Equation

For small perturbations of an inviscid, incompressible, stratified, Boussinesq shear flow, we have the following set of linear equations:

\[ u_t + u u_x + v u_y + w u_z = -p' \]  
\[ v_t + u v_x + v v_y + w v_z = -p' \]  
\[ w_t + u w_x + v w_y + w w_z = -g \rho' - p' \]  
\[ u' + v' + w' = 0 \]  \( \text{and} \)  
\[ p' = 0 \]  \( \text{where} \) (u', v', w') is the perturbation velocity in Cartesian coordinates. Subscripts x, y, z, and t denote differentiation. U(z) = (U(z), V(z), 0), and U(z) and V(z) represent mean velocity components in the zonal and meridional directions, respectively. The undisturbed mean density profile is \( \rho_0 \), \( \rho' = (\rho - \rho_0) / \rho_0 \), and \( p' \) is the perturbation pressure, also divided by \( \rho_0 \).

Perturbation eigenfunctions \( \hat{u}(z), \hat{v}(z), \hat{p}(z) \) can be expressed in terms of \( \hat{w}(z) \) using the polarization relations:

\[ \hat{u} = -i \frac{1}{k^2 \sigma} (U_z - k V_z) \hat{w} + i \frac{k}{k^2} \hat{w}_z \]  
\[ \hat{v} = i \frac{k}{k^2 \sigma} (U_z - k V_z) \hat{w} + i \frac{1}{k^2} \hat{w}_z \]  
\[ \hat{\rho} = \frac{N^2}{g \sigma} \]  
\[ \hat{p} = i \frac{1}{k^2} (V_z + k U_z) \hat{w} + i \frac{1}{k^2} \hat{w}_z \]  \( \text{where} \) \( N \) is the local buoyancy frequency, defined by \( N^2(z) = -g \rho' / \rho_0 \). The variable \( \sigma = \omega - \mathbf{U} \cdot \mathbf{K} \) represents the Doppler-shifted complex frequency.

Equation (A2) is now obtained by substituting (A6) and (A7) into (A4). The perturbation kinetic energy equation (3) is obtained by taking the scalar product of \( ((A1), (A2), (A3)) \) with \( (u', v', w') \), averaging over \( x \) and \( y \) and using (A4) to simplify the pressure term.

\[ \hat{w}_z = \text{im} \hat{w} \]  \( \text{at} \) \( z = -D \) \( \text{where} \)

\[ m = \pm \sqrt{\frac{N^2}{(U - c)^2} - \frac{\hat{U}_z}{(U - c) - k^2} \frac{1}{2}} \]  \( \text{The sign is chosen so that the vertical kinetic energy flux} \) \( \text{at the lower boundary (averaged over one horizontal wavelength) is directed downward, that is,} \)

\[ p' w' = \frac{|\hat{w}|^2}{2 \pi} \text{Re}(m(U - c)) < 0 \]  \( z = -D \)

This is equivalent to

\[ \text{Re}(m(U - c)) < 0 \]  \( z = -D \)  \( \text{B3} \)

With rigid boundary conditions imposed at both horizontal boundaries, analyses are simplified by certain properties of the solutions of the Taylor-Goldstein equation. For example, if there exists a solution with eigenvalue c and eigenfunction \( \hat{w} \), then eigenvalue \( c^* \) and eigenfunction \( \hat{w}^* \) are also solutions [e.g., Hazel, 1972]. The Miles-Howard theorem states that if Richardson number is greater than 1/4 everywhere in the flow, then the flow field is linearly stable, that is, infinitesimal disturbances will not grow exponentially with time [Miles, 1961; Howard, 1961]. Howard's semicircle theorem provides a useful bound on the complex phase velocity of an unstable mode [Hazel, 1972]. However, it is not immediately clear what the effects of radiation boundary condition are on these properties, and we consider those effects further here.

When the impermeability condition \( \hat{w} = 0 \) is imposed at both boundaries and an eigenvalue solution exists with complex phase speed \( c \), then there is also a mode with phase speed \( c^* \); that is, every growing mode is accompanied by a decaying mode and vice versa [e.g., Hazel, 1972]. Here we will show that the radiation condition upsets this symmetry. Assume we have a solution of (2) and boundary conditions with the complex phase speed \( c_1 \), and the corresponding eigenfunction is \( \hat{w}_1 \). Let us investigate if eigenvalue \( c_1 \) and eigenfunction \( \hat{w}_2 = \hat{w}_1^* \) will satisfy the radiation condition at the lower boundary. Specifically, we want to investigate whether the vertical flux of perturbation kinetic energy \( \hat{p}_2 \hat{w}_2^* \) is downward at the lower boundary \( z = -D \). From the polarization relation (A9),

\[ \hat{p}_2 = -\hat{p}_1^* \]  \( \text{B4} \)

Therefore

\[ \hat{p}_2 \hat{w}_2^* = \frac{1}{2} \text{Re}(\hat{p}_2 \hat{w}_2^*) = \frac{1}{2} \text{Re}(-\hat{p}_1^* \hat{w}_1) = -\hat{p}_1 \hat{w}_1^* > 0 \]

The radiation boundary condition cannot be satisfied with the complex conjugate solution; that is, the complex conjugate symmetry is no longer valid when the radiation condition is applied.

The Howard semicircle theorem for the case with a radiation condition at the lower boundary and an im-

Appendix B: Radiation Boundary Condition and Its Effects

So that waves generated at the shear layer can propagate downward into the thermocline, a radiation boundary condition is imposed at the lower boundary:
The permeability condition at the upper boundary becomes 
\[ \text{[De Bass and Driedonks, 1985]} \]
\[ [c_i - \frac{1}{2} (\bar{U}_{\text{min}} + \bar{U}_{\text{max}})]^2 + c_i^2 \leq \frac{1}{4} (\bar{U}_{\text{min}} - \bar{U}_{\text{max}})^2 - B^2 \]  
where
\[ B^2 = \frac{\int_{-D}^{0} N^2 |F|^2 \, dz}{\int_{D}^{0} Q \, dz} \]  
(\text{B5})

\[ F = \frac{\dot{w}}{U - c} \]  
and \[ Q = |F|^2 / \kappa^2 |F|^2 > 0. \]  
The final term on the right-hand side of \( \text{(B5)} \) represents the modification due to the radiation condition. The new term has the effect of shrinking the semicircle within which instability is possible. Unfortunately, the new term involves the eigenfunction, so it cannot be used to sharpen the a priori bounds on unstable eigenvalues. However, we can conclude that the original semicircle theorem remains valid when the radiation condition is applied.

It was shown by Miles [1961] and Howard [1961] that, under rigid boundary conditions, the flow is stable to disturbances of small amplitude if the Richardson number \( \text{Ri} = N^2 / (U_z^2 + V_z^2) \) is everywhere larger than the critical value 1/4. For the three-dimensional case considered here, the effective Richardson number \( \text{Ri} = N^2 / U_z^2 \) is computed using the vertical derivative of the component of the basic flow parallel to the horizontal wavenumber. \( \text{Ri} \) is always larger than or equal to the Richardson number \( \text{Ri} \) [e.g., Booker and Bretherton, 1967]. For a specific direction \( \theta \), the Miles-Howard theorem assures us that if \( \text{Ri} \) is greater than 1/4 everywhere, then the flow is stable to disturbances that propagate in that direction. Here we show that the Miles-Howard theorem remains valid in the presence of a radiation boundary condition; that is, Richardson number \( \text{Ri} \) smaller than 1/4 at some depth is still a necessary condition for shear instability.

Following Howard [1961], let us define a new variable \( \phi \) by
\[ \phi = \frac{\dot{w}}{\sqrt{U - c}} \]  
(\text{B7})

With this substitution, the Taylor-Goldstein equation becomes
\[ \{ (U - c) \phi_z \}_z - \{ \kappa^2 (U - c) + \frac{1}{2} U_z^2 + \frac{1}{2} U_z^2 - N^2 \} \phi = 0 \]  
(\text{B8})

The boundary conditions are
\[ \phi = 0 \quad z = 0 \]  
\[ \phi_z = i \kappa \phi - \frac{U_z}{2(U - c)} \phi \quad z = -D \]  
(\text{B9})

Now multiply \( \text{(B8)} \) by the complex conjugate of \( \phi \), integrate from \( z = -D \) to \( z = 0 \), and take the imaginary part
\[ c_i \int_{-D}^{0} \left\{ \frac{N^2 - \frac{1}{2} \bar{U}_z^2}{|U - c|^2} |\phi|^2 + |\phi_z|^2 + \kappa^2 |\phi|^2 \right\} \, dz = \]  
\[ \text{Re} \{ m(U(-D) - c)|\phi(-D)|^2 \} \]  
(\text{B11})

If \( N^2 > \frac{1}{2} \bar{U}_z^2 \) everywhere in the basic flow, then the integral on the left-hand side is positive, while the right-hand side of the equation is negative due to the radiation boundary condition \( \text{(B3)} \). We thus conclude that \( c_i \) must be negative. Therefore linear stability of the flow field is guaranteed if the Richardson number \( \text{Ri} > 1/4 \) everywhere in the flow, just as in the case of impermeable boundaries. Note that since \( \text{Ri} \geq \text{Ri} \), the stronger condition \( \text{Ri} > 1/4 \) guarantees stability for all \( \theta \).

The effects of the radiation condition on the validity of the Miles-Howard and semicircle theorems are intuitively reasonable. Because the radiation condition only allows perturbation energy to be removed from the domain, its presence tends to stabilize the flow. Therefore, in regions of parameter space where the flow is demonstrably stable in the presence of rigid boundaries, the flow will remain stable when outgoing radiation is allowed.

Acknowledgments. We are grateful to Murray Levine, Doug Caldwell, Eric Kunze, and John Allen for helpful discussions. We obtained the surface current mooring data from Michael McPhaden of NOAA Pacific Marine Environmental Laboratory. This work was funded by the National Science Foundation (OCE-8816098, OCE-9314396, and OCE-9521359).

References


Lien, R.-C., M. McPhaden, and M. C. Gregg, High-frequency internal waves at 0ø, 140øW and their possible relationship to deep-cycle turbulence, J. Phys. Oceanogr., 26, 1409–1425, 1996.


Lien, R.-C., M. McPhaden, and M. C. Gregg, High-frequency internal waves at 0ø, 140øW and their possible relationship to deep-cycle turbulence, J. Phys. Oceanogr., 26, 581–600, 1996.


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(Received July 16, 1997; revised December 9, 1997; accepted January 16, 1998.)