Anisotropy of turbulence in stably stratified mixing layers

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Direct numerical simulations of turbulence resulting from Kelvin–Helmholtz instability in stably stratified shear flow are used to study sources of anisotropy in various spectral ranges. The set of simulations includes various values of the initial Richardson and Reynolds numbers, as well as Prandtl numbers ranging from 1 to 7. We demonstrate that small-scale anisotropy is determined almost entirely by the spectral separation between the small scales and the larger scales on which background shear and stratification act, as quantified by the buoyancy Reynolds number. Extrapolation of our results suggests that the dissipation range becomes isotropic at buoyancy Reynolds numbers of order $10^5$, although we cannot rule out the possibility that small-scale anisotropy persists at arbitrarily high Reynolds numbers, as some investigators have suggested. Correlation-coefficient spectra reveal the existence of anisotropic flux reversals in the dissipation subrange whose magnitude decreases with increasing Reynolds number. The scalar concentration field tends to be more anisotropic than the velocity field. Estimates of the dissipation rates of kinetic energy and scalar variance based on the assumption of isotropy are shown to be accurate for buoyancy Reynolds numbers greater than $O(10^5)$. Such estimates are therefore reliable for use in the interpretation of most geophysical turbulence data, but may give misleading results when applied to smaller-scale flows.

I. INTRODUCTION

This paper is one of a sequence reporting results from direct numerical simulations of turbulence arising from the dynamic instability of stably stratified shear layers. The work is motivated by the need to properly interpret measurements of turbulent events in the Earth’s oceans; however, the results have important implications for a wide range of turbulent flows. Our focus in the present paper is on issues of local isotropy, the tendency of the smallest scales of motion to be isotropic despite the anisotropy of the large scales.

In the view of turbulence that originated with Kolmogorov, it is thought that the cascade of energy from large to small scales of motion is accompanied by a loss of information about the geometry of the large scales. In the limit of high Reynolds number, the smallest eddies have a structure that is independent of the large-scale flow, and is in that sense universal. In other words, the small-scale structure depends only on the equations of motion, and not on initial or boundary conditions. In the absence of body forces, the Navier–Stokes equations are isotropic, i.e., preferred orientations enter only through initial and boundary conditions. Therefore, a universal state dependent only on the equations themselves should exhibit the property of isotropy.

Turbulence occurring in nature is usually influenced to some degree by density stratification and background shear. The addition of buoyancy forces introduces a preference for the vertical direction (the direction of the local gravitational field) into the equations. However, it is believed that buoyancy acts preferentially upon motions larger than the Ozmidov scale (to be defined in the following), and that scales much smaller than this are unaffected. We therefore expect that, at high Reynolds number, motions on sufficiently small spatial scales will be isotropic despite the action of buoyancy forces. In sheared turbulence, the background shear induces anisotropy in the large eddies. As in the case of buoyancy, however, the effect is expected to become negligible at sufficiently small scales.

Isotropy of the small scales is not proven. While many experiments have shown that one or more turbulence statistics become consistent with local isotropy at large Reynolds numbers (e.g., the wind tunnel experiments of Saddoughi and Veeravalli), the hypothesis asserts that every statistic is consistent with local isotropy. To the contrary, Durbin and Speziale have argued from theory that the small scales cannot be isotropic at any Reynolds number in the presence of a steady background strain. Both laboratory experiments and geophysical measurements (in which the Reynolds number is high) have revealed distinct departures from isotropy. Stability analyses of many flows have shown that steady, large-scale structures may inject energy directly into the smallest scales of motion regardless of the Reynolds number. Yeung and colleagues have conducted numerical explorations of the mechanisms through which large-scale anisotropy may be transferred to the smallest scales, even at high Reynolds number. So, although the likelihood that local isotropy holds at sufficiently large Reynolds number is widely accepted, we cannot yet discount the possibility that some forms of small-scale anisotropy persist at all Reynolds num-

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bers of practical interest, or even to the limit of infinite Reynolds number.

The importance of isotropy in the small scales is not limited to our conceptual understanding of turbulence. The molecular dissipation of kinetic energy, as well as that of scalar fluctuations, takes place primarily at the smallest scales of motion, in the so-called dissipation range of the wave number spectrum. The rates at which these crucial processes occur are very difficult to measure, as they require the simultaneous measurement of numerous spatial derivatives. The problem becomes greatly simplified, however, if one assumes that motions in the dissipation range are isotropic. In that case, the number of simultaneous measurements needed to infer dissipation rates is greatly reduced. But even if the hypothesis of local isotropy is valid, the smallest scales of motion are expected to be isotropic only as the Reynolds number increases to infinity. In naturally occurring flows, we need to account for the consequences of finite Reynolds number. For a given large-scale flow geometry at a given Reynolds number, how anisotropic are the smallest scales, i.e., what are the uncertainties associated with the assumption of isotropy in the dissipation range? Is local isotropy really a function of Reynolds number only? What are the physical mechanisms through which information about the geometry of the large scales is transmitted down the spectrum?

The present work was motivated by the need to properly interpret observations of temperature and velocity microstructure in the ocean. Such measurements almost always have the form of one-dimensional profiles. Dissipation rates are inferred from this limited information using isotropic approximations, even though the Reynolds number is finite and both background shear and density stratification are typically present. Using measurements of flow over a sill, Garrett et al. have shown that certain properties of the inertial subrange are consistent with isotropy for values of the buoyancy Reynolds number, \( R_b \), (to be defined in the following) in excess of 200. Itsweire et al. tested isotropic approximations for dissipation range statistics using data from numerical simulations of turbulence with uniform background shear and stratification, and found large errors for \( R_b < 10^2 \). Here, we extend the analyses of Itsweire et al. to more realistic flow geometries and consider the physics of local anisotropy in greater detail.

Our primary tool in these analyses is direct numerical simulations (DNS) of breaking Kelvin–Helmholtz (KH) billows. KH billows are commonly observed in geophysical fluid systems, and are an important source of geophysical turbulence. They provide a useful model for turbulent events in the ocean interior, and for sheared, stratified turbulence in general. KH billows occur when a vertically sheared, horizontal flow coexists with stable density stratification such that the gradient Richardson number (a measure of the strength of the stratification relative to the shear) is smaller than a critical value. Billows take the form of line vortices aligned horizontally and perpendicular to the mean flow, and result in an increase in the gravitational potential energy of the fluid. As the billows reach large amplitude, transition to turbulence is initiated via a combination of convective and shear-driven secondary instabilities. Even after the onset of turbulence, the large eddies reflect the structure of the original billows. The result is a flow which is wave-like on the largest scales but highly chaotic on smaller scales. The disturbance then decays, leaving a field of weak gravity waves propagating on a stable, parallel shear flow.

Our ultimate goal is to assess the validity of various isotropic formulas over a range of Reynolds numbers. As preparation for this, we first examine the relationship between the different scales of motion during each phase of the flow evolution, paying particular attention to the degree to which the anisotropic structure of the large scales is reflected in the small scales. Our dataset consists of eight simulations in which both the initial flow configuration and the Prandtl number of the fluid are varied. The Reynolds number varies in time as turbulence grows and subsides, and is also governed by the initial conditions that are specific to each simulation. The resulting dataset spans a wide range of flow regimes, with buoyancy Reynolds numbers as large as 200. Our analyses enable us to draw general conclusions about the physics of anisotropy in sheared, stratified turbulence, as well as assessing the validity of popular isotropic approximations.

At all Reynolds numbers, signatures of anisotropy are detected throughout the spectrum. When turbulence is most vigorous, however, small-scale anisotropy is detectable only by the most sensitive tests. A significant finding is that, for a wide variety of turbulent flow states, quantities associated with anisotropy in the dissipation scales can be predicted quite precisely based only on knowledge of a suitably defined Reynolds number. The isotropic approximations vary in accuracy, but are generally valid for buoyancy Reynolds numbers in excess of \( O(10^5) \), which encompasses the large majority of ocean turbulence measurements to date (e.g., Moum).

Section II of this paper contains a brief review of the numerical model and a description of the diagnostic methods used in the present analyses. In Sec. III, we give an overview of the life cycle of turbulence originating from KH instability. In Sec. IV, we describe the geometry of the large-scale (i.e., energy-containing) flow structures. Section V is devoted to measures of anisotropy in the dissipation range, as revealed by the statistical characteristics of velocity gradients. We examine the physical mechanisms responsible for anisotropy in these small-scale motions, the manner in which anisotropy vanishes as the Reynolds number increases, and the impact of anisotropy on estimates of the dissipation rate for turbulent kinetic energy. We also show that our most turbulent states exhibit a distinct inertial subrange. In Sec. VI, we carry out a similar analysis of anisotropy in spatial gradients of density. Results are summarized in Sec. VII.

II. METHODOLOGY

A. DNS methods

The numerical methods employed to generate the present dataset are described in detail in the companion
paper; here, we provide a brief summary. Our mathematical model employs the Boussinesq equations for velocity, density, and pressure in a nonrotating physical space measured by the Cartesian coordinates $x$, $y$, and $z$:}

$$
\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times (\nabla \times \mathbf{u}) - \nabla \Pi + g \dot{k} + \nu \nabla^2 \mathbf{u};
$$

(1)

$$
\Pi = \frac{p}{\rho_0} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}.
$$

(2)

Here, $p$ is the pressure and $\rho_0$ is a constant characteristic density. The thermodynamic variable $\theta$ represents the fractional specific volume deviation, or minus the fractional density deviation, i.e., $\theta = -(\rho - \rho_0)/\rho_0$. In a fluid where density is controlled only by temperature, $\theta$ is proportional to the temperature deviation (with proportionality constant equal to the thermal expansion coefficient). The gravitational acceleration $g$ has the value 9.8 m/s$^2$, and $\dot{k}$ is the vertical unit vector. Viscous effects are represented by the usual Laplacian operator, with kinematic viscosity $\nu = 1.0 \times 10^{-6}$ m$^2$/s.

The augmented pressure field $\Pi$ is specified implicitly by the incompressibility condition

$$
\nabla \cdot \mathbf{u} = 0,
$$

(3)

and the scalar $\theta$ evolves in accordance with

$$
\frac{\partial \theta}{\partial t} = -\mathbf{u} \cdot \nabla \theta + \kappa \nabla^2 \theta,
$$

(4)

in which $\kappa$ represents the molecular diffusivity of $\theta$.

We assume periodicity in the horizontal dimensions:

$$
f(x + L_x, y, z) = f(x, y + L_y, z) = f(x, y, z),
$$

(5)

in which $f$ is any solution field and the periodicity intervals $L_x$ and $L_y$ are constants. At the upper and lower boundaries ($z = \pm \frac{1}{2} L_z$), we impose an impermeability condition on the vertical velocity:

$$
w|_{z = \pm (1/2)L_z} = 0,
$$

(6)

and zero-flux conditions on the horizontal velocity components $u$ and $v$ and on $\theta$:

$$
\frac{\partial u}{\partial z} \bigg|_{z = \pm (1/2)L_z} = \frac{\partial v}{\partial z} \bigg|_{z = \pm (1/2)L_z} = \frac{\partial \theta}{\partial z} \bigg|_{z = \pm (1/2)L_z} = 0.
$$

(7)

These imply a condition on $\Pi$ at the upper and lower boundaries:

$$
\left[ \frac{\partial \Pi}{\partial z} - g \theta \right]_{z = \pm (1/2)L_z} = 0.
$$

(8)

The model is initialized with a parallel flow in which shear and stratification are concentrated in the shear layer, a horizontal layer surrounding the plane $z = 0$:

$$
\tilde{u}(z) = \frac{u_0}{2} \tanh \frac{2z}{h_0}, \quad \tilde{\theta}(z) = \frac{\theta_0}{2} \tanh \frac{2z}{h_0}.
$$

(9)

The constants $h_0$, $u_0$, and $\theta_0$ represent the initial thickness of the shear layer and the changes in velocity and density across it. These constants can be combined with the fluid parameters $\nu$ and $\kappa$ and the geophysical parameter $g$ to form three dimensionless groups whose values determine the stability of the flow at $t = 0$:

$$
\text{Re}_0 = \frac{u_0 h_0}{\nu}, \quad \text{Ri}_0 = \frac{g \theta_0 h_0}{\nu^2}, \quad \text{Pr} = \frac{\nu}{\kappa}.
$$

(10)

The initial macroscale Reynolds number, $\text{Re}_0$, expresses the relative importance of viscous effects. In the present simulations, $\text{Re}_0$ is of order a few thousand, large enough that the initial instability is nearly inviscid. The bulk Richardson number, $\text{Ri}_0$, quantifies the relative importance of shear and stratification. If $\text{Ri}_0 < 0.25$, the initial mean flow possesses unstable normal modes. The Prandtl number, $\text{Pr}$, is the ratio of the diffusivities of momentum and density. In order to obtain a fully turbulent flow efficiently, we add to the initial mean profiles a perturbation field designed to excite the most-unstable primary and secondary instabilities. Details of the initial perturbation are given in the companion paper.

Care is taken to ensure that the initial perturbation is weak enough to obey linear physics. In that case, the precise form of the perturbation has little effect on the statistical quantities of interest here.

Because the horizontal boundary conditions are periodic, discretization of the horizontal differential operators is accomplished efficiently in Fourier space using fast Fourier transforms. In the vertical direction, we employ second-order, centered finite differences, in order to retain flexibility in the choice of upper and lower boundary conditions. The fields are stepped forward in time using a second-order Adams–Bashforth method. Implicit time stepping of the viscous operators has turned out to be unnecessary, as the time step is limited by the advective terms. For accuracy and numerical stability, we limit the time step in accordance with

$$
\Delta t < 0.12 \min_{i=1,3} \left( \frac{\Delta x_i}{U_i^+} \right),
$$

(11)

in which $U_i^+$ is the maximum speed in the $x_i$ coordinate direction. At each time step, the augmented pressure field is obtained as the solution of a Helmholtz equation which forces the flow to be nondivergent at the next timestep.

A well-resolved DNS model must have grid spacing no greater than a few ($3–6$) times the Kolmogorov length scale $L_K = (\nu^3/\epsilon)^{1/4}$ in which $\epsilon$ is the volume-averaged kinetic energy dissipation rate (to be defined in the following). For flows with $\text{Pr}>1$, the Kolmogorov scale is replaced by the Batchelor scale: $L_B = L_K / \text{Pr}^{1/2}$. In order to take advantage of this information, one must be able to estimate the maximum value of $\epsilon$ in advance. We do this by employing the scaling $\epsilon^+ = c u_0^+ h_0$. Through trial and error, we have established the value $6.5 \times 10^{-4}$ for the proportionality constant $c$. The grid spacing is then set to $2.5 L_B^+$. The smallest Batchelor scale $(\nu^3/\epsilon^+)^{1/6} \text{Pr}^{-1/2}$. The present version of the model uses isotropic grid resolution, i.e., $\Delta x = \Delta y = \Delta z$. (The largest of the simulations described here employed array sizes of $512 \times 64 \times 256$, and required 800
TABLE I. Parameter values describing a sequence of eight simulations of breaking Kelvin–Helmholtz billows.

<table>
<thead>
<tr>
<th>Run</th>
<th>N_x</th>
<th>N_y</th>
<th>N_z</th>
<th>L_x (m)</th>
<th>L_y (m)</th>
<th>L_z (m)</th>
<th>Re_0</th>
<th>R_i_0</th>
<th>P_r</th>
<th>Ri_0</th>
<th>Re_0</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>R0P1</td>
<td>256</td>
<td>64</td>
<td>128</td>
<td>3.29</td>
<td>0.82</td>
<td>1.63</td>
<td>12865</td>
<td>0.08</td>
<td>1965</td>
<td>1967</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>R16P1</td>
<td>512</td>
<td>64</td>
<td>256</td>
<td>5.24</td>
<td>0.82</td>
<td>2.60</td>
<td>4978</td>
<td>0.12</td>
<td>4978</td>
<td>1967</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>R0P4</td>
<td>256</td>
<td>64</td>
<td>128</td>
<td>3.29</td>
<td>0.82</td>
<td>1.63</td>
<td>12865</td>
<td>0.16</td>
<td>1967</td>
<td>1967</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>R10P1</td>
<td>512</td>
<td>64</td>
<td>256</td>
<td>3.29</td>
<td>0.82</td>
<td>1.63</td>
<td>4978</td>
<td>0.12</td>
<td>4978</td>
<td>1967</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>R0P3</td>
<td>512</td>
<td>64</td>
<td>256</td>
<td>3.29</td>
<td>0.82</td>
<td>1.63</td>
<td>12865</td>
<td>0.08</td>
<td>1967</td>
<td>1967</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>R0P4</td>
<td>512</td>
<td>64</td>
<td>256</td>
<td>3.29</td>
<td>0.82</td>
<td>1.63</td>
<td>4978</td>
<td>0.12</td>
<td>4978</td>
<td>1967</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>R0P5</td>
<td>512</td>
<td>64</td>
<td>256</td>
<td>3.29</td>
<td>0.82</td>
<td>1.63</td>
<td>12865</td>
<td>0.08</td>
<td>1967</td>
<td>1967</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

The time scale Mbytes of core memory and 100–200 h of CPU time on a 32-node Connection Machine CM5.) Spectra confirming the accuracy of the numerics, along with comparisons with laboratory experiments, are provided elsewhere.1,2

The dataset employed here consists of eight simulations, designed to reveal the effects of changing R_i_0, R_e_0, and P_r. In order to adequately sample the regime of KH instability while maintaining accuracy, R_i_0 was set to 0.08, 0.12, and 0.16. Prandtl numbers of 1, 4, and 7 were employed. The choice of R_e_0 was strongly constrained by the need to maintain adequate resolution of the small scales; values ranged from 1354 to 4978. As the evolution progressed, both the bulk Richardson and Reynolds numbers increased in proportion to the increasing depth of the shear layer.2 For all runs, the time scale h_0/u_0 was set to 28.28 s. Parameter values for all eight simulations are summarized in Table I.

B. Diagnostic methods

For the present analyses, we require statistical quantities which give the clearest possible picture of the structure of the turbulence, the structure of the large-scale environment in which the turbulence evolves, and the relationship between the two. We begin with a definition of the averaging volume. Our simulated turbulence is of infinite horizontal extent (via periodic boundary conditions), but is confined in the vertical to a finite layer. Outside that layer are regions of laminar flow that extend to the upper and lower boundaries of the computational domain. The locations of the upper and lower boundaries of the turbulent layer, z_U and z_L, are to some degree arbitrary. Our objective is to define these locations in a maximally straightforward manner consistent with the need to exclude the outer, laminar layers. After some experimentation, we have defined z_U(x,y,t) and z_L(x,y,t) to be the surfaces upon which \( \theta \) takes the values \( c \theta_0/2 \) and \( -c \theta_0/2 \), respectively, where \( c = \tanh(1) \). The value of \( c \) is large enough to ensure that z_U and z_L remain single-valued functions of horizontal position. The volume \( z_L < z < z_U \) is referred to as \( V_z \). The efficacy of \( V_z \) in isolating the turbulent region of the computational domain is demonstrated below (cf. Fig. 3 and the accompanying discussion). In this paper, volume averages are taken over \( V_z \), with a few exceptions to be specified explicitly.

We also require values for the shear and stratification characteristic of the large-scale flow at any given time. The background scalar gradient, \( \theta_z \), is defined as the vertical derivative of the horizontally averaged scalar concentration, \( \theta(z,t) \), averaged vertically over the layer in which \( -c \theta_0/2 < \theta < c \theta_0/2 \). (Subscripts preceded by commas indicate partial differentiation.) This layer is related to \( V_z \), but is defined in terms of horizontally averaged quantities, so that its boundaries are independent of \( x \) and \( y \). Background stratification is then characterized by the buoyancy frequency, \( N(x,y) = \sqrt{g \nabla \theta_z} \). In similar fashion, we define the background shear, \( S \), to be the average of \( \bar{u}_z \) over the layer in which the horizontally averaged velocity \( \bar{u}(z,t) \) satisfies \( -c u_0/2 < \bar{u} < c u_0/2 \). Note that both \( S \) and \( \nabla \theta_z \) are positive for the flows considered here.

To characterize relationships among \( S, N \), and the turbulence statistics at any given time, we choose from several scalar quantities that are used commonly in the interpretation of turbulence data. These are the buoyancy Reynolds number, the shear Reynolds number, the shear number, and the Cox number. The buoyancy Reynolds number is given by

\[
R_b = \frac{\langle e \rangle}{\nu N^2}.
\]

Here, \( \langle e \rangle \) represents the average over \( V_z \) of the kinetic energy dissipation rate, \( \varepsilon = 2 \nu s_{ij} s_{ij} \), where
is the strain tensor. Primes denote fluctuations about the horizontally averaged fields.

A second measure of spectral range is the shear Reynolds number:

\[ R_s = \frac{\langle \epsilon \rangle}{\nu^2}. \]  

The shear Reynolds number is equal to \((L_C/L_K)^{4/3}\), where \(L_K\) is the Kolmogorov length scale, \((\nu^3/\langle \epsilon \rangle)^{1/4}\), and \(L_C = (\langle \epsilon \rangle/S^3)^{1/2}\) is referred to here as the Corrsin scale. \(^27\) an estimate of the smallest scale at which eddies are strongly deformed by the mean shear. In similar fashion, \(R_b\) equals \((L_Q/L_K)^{4/3}\), where \(L_Q = (\langle \epsilon \rangle/N^3)^{1/2}\) is the Ozmidov scale, a measure of the smallest scale that experiences significant deformation by the mean density gradient.

\(R_s\) is the inverse square of the quantity referred to by Saddoughi and Veeravalli\(^6\) as \(S^\#\), the latter being the large-scale shear scaled by the strain rate of Kolmogorov eddies, \(\sqrt{\langle \epsilon \rangle}/\nu\). A useful measure of the strength of the large-scale shear relative to the strain of the energy-containing scales is the shear number:

\[ S^\# = S \frac{q^2}{\langle \epsilon \rangle}, \]  

in which \(q^2 = \langle u'_i u'_i \rangle\) is twice the specific kinetic energy of the velocity fluctuations.

Finally, we include the Cox number,

\[ \text{Cox} = \frac{\langle |\nabla \theta'|^2 \rangle}{\langle \theta'_c \rangle^2}. \]  

\(R_s\) may be interpreted as the ratio of kinetic energy dissipation due to turbulence to that accomplished by the mean flow. The Cox number may be interpreted in analogous fashion; it is the ratio of potential energy dissipation by turbulence to that accomplished by the mean flow.

A convenient way to characterize anisotropy is through the use of symmetric, traceless tensors whose elements vanish in isotropic flow. Anisotropy in the energy-containing scales is quantified using the velocity anisotropy tensor:

\[ b_{ij} = \frac{\langle u'_i u'_j \rangle}{\langle u'_k u'_k \rangle} \delta_{ij} / 3, \]  

in which \(\delta\) is the Kronecker delta and summation over repeated indices is implied. With some exceptions (to be discussed in due course), volume averages are taken over \(V_s\) and are indicated by angle brackets. Again, primes indicate fluctuations about the mean (i.e., horizontally averaged) velocity profile. (The mean flow is removed before calculating \(b_{ij}\) because, otherwise, it dominates the tensor so completely that the properties of the velocity fluctuations cannot be discerned. In other anisotropy tensors, to be described in the following, removal of the mean profile is unnecessary.) The structure of the velocity anisotropy tensor \(b_{ij}\) is conveniently represented using the two invariants

\[ \Pi_b = b_{ij} b_{ji}, \quad \Pi_b = b_{ij} b_{jk} b_{ki}. \]  

Because \(b\) is traceless, \(\Pi_b\) and \(\Pi_b\) are proportional to the second and third principal invariants of \(b\), respectively. \(^28\) Here, we follow previous terminology (e.g., Lumley and Newman\(^29\)) by referring to \(\Pi_b\) and \(\Pi_b\) as the second and third invariants, respectively, of \(b\). \(\Pi_b\) is positive definite, and represents the degree of anisotropy in the velocity field. \(\Pi_b\) can take either sign; its value contains information about the nature of the anisotropy. When \(\Pi_b > 0\) the flow tends to be dominated by one Cartesian component that is much larger than the other two, whereas when \(\Pi_b < 0\), two components dominate. We refer to these two classes of anisotropy as "prolate" and "oblate," respectively, in analogy with the usage of those terms to classify ellipsoids. (Other authors have used the terms "rod-like" and "disk-like" for the same concepts.\(^30\)) For a given value of \(\Pi_b\), \(\Pi_b\) is bounded from above by the two-component limit \(\Pi_b = 2/3 + 2\Pi_b\), and from below by the axisymmetric limit \(\Pi_b = 6^{1/3}|\Pi_b|^{1/3}\) (cf. Fig. 6). Lumley and Newman\(^29\) provide further details on the physical interpretations of \(\Pi_b\) and \(\Pi_b\). George and Hussein\(^31\) provide a detailed discussion of the axisymmetric case.

Anisotropy in the vorticity-carrying motions is represented by the vorticity anisotropy tensor,

\[ v_{ij} = \frac{(\omega_i \omega_j)}{(\omega_k \omega_k)} - \frac{\delta_{ij}}{3}, \]  

and its invariants \(\Pi_v = v_{ij} v_{ji}\) and \(\Pi_v = v_{ij} v_{jk} v_{ki}\). Here, \(\omega\) represents the total vorticity, \(\omega = \nabla \times u\). The interpretation of the invariants is similar to that given in Sec. II A for the invariants of \(b\), with one important exception: in two-dimensional flow, only a single component of the vorticity is nonzero. That flow configuration therefore corresponds to the single-component limit, \((\Pi_v, \Pi_v) = (2/3, 2/9)\).

Anisotropy of the scalar gradient field is represented in similar fashion by the tensor

\[ G_{ij} = \frac{(\theta_{ij} \theta_{ij})}{(\theta_k \theta_k)} - \frac{\delta_{ij}}{3}, \]  

and its invariants \(\Pi_G = G_{ij} G_{ji}\) and \(\Pi_G = G_{ij} G_{jk} G_{ki}\).

An anisotropy tensor pertaining to the dissipation rate is defined in a slightly different manner:

\[ d_{ij} = \frac{(u_i \delta_{ij} u_j)}{(u_k \delta_{jk} u_l)} - \frac{\delta_{ij}}{3}. \]  

Again, commas before subscripted indices indicate partial differentiation. The invariants of \(d\) are denoted \(\Pi_d\) and \(\Pi_d\).

An alternative method of quantifying anisotropy, often used in the interpretation of laboratory data,\(^31,32\) is via the ratios

\[ K_1 = \frac{\langle u_x'^2 \rangle}{\langle u_x'^2 \rangle}; \quad K_2 = \frac{\langle w_x'^2 \rangle}{\langle w_x'^2 \rangle}; \quad K_3 = \frac{\langle u_y'^2 \rangle}{\langle u_y'^2 \rangle}; \quad K_4 = \frac{\langle w_y'^2 \rangle}{\langle w_y'^2 \rangle}. \]  

In isotropic flow, each of these ratios is equal to unity.
The most sensitive tests for anisotropy employ the one-dimensional streamwise cospectra $E_{ij}(k_1)$, in which $k_1$ is the streamwise wave number. $E_{ij}$ is normalized such that 

$$
\int_0^\infty E_{ij}(k_1)dk_1 = \bar{u_i}u_j,
$$

where the overbar indicates a streamwise average. The corresponding correlation-coefficient spectra are given by 

$$
C_{ij} = \frac{E_{ij}}{\sqrt{E_{ii}E_{jj}}}.
$$

(23)

In the foregoing equation only, repeated indices are not summed. To compensate for the distinction between longitudinal and transverse one-dimensional spectra, $E_{11}$ can be replaced by its isotropic transverse equivalent: 

$$
E'_{11} = \frac{1}{2} \left( E_{11} - k_1\frac{dE_{11}}{dk_1} \right).
$$

(25)

In a region of the spectrum where motions are isotropic, $E'_{11} = E_{22} = E_{33}$ and $E_{12} = E_{23} = E_{13} = C_{12} = C_{23} = C_{13} = 0$.

III. OVERVIEW OF TURBULENCE EVOLUTION

In this section, we give an overview of turbulence evolution in representative simulated turbulent events. The KH life cycle includes the initial growth and pairing of the two-dimensional primary billows, breaking and the transition to turbulence, and an extended period of slow decay during which the flow relaxes to a stable, parallel configuration.

We begin with a direct visualization of enstrophy isosurfaces in physical space (Fig. 1). At $t=1131$ s [Fig. 1(a)], two KH billows have grown to large amplitude and are in the process of merging. By $t=3111$ s [Fig. 1(b)], the transition process is complete. While the range of scales is necessarily limited by computer size, the small scales are in many respects characteristic of fully developed turbulence (also see Fig. 3). The spatial signatures of the three-dimensional secondary instabilities that drive the transition remain visible as streaks in the braid regions at the left and right of the domain, but the vorticity field within the billow is highly disordered. Figure 1(c) typifies the early part of the decay phase. Turbulence is well-developed, with horseshoe-like...
vortices visible near the top of the turbulent layer. By the end of the simulation [Fig. 1(d)], turbulence has decayed almost completely, leaving behind a dynamically stable, parallel shear flow perturbed by laminar gravity waves.

Next, we examine the volume-averaged perturbation kinetic energies associated with the three Cartesian components of the velocity [Fig. 2(a)], along with the buoyancy Reynolds number, the shear Reynolds number, and the Cox number [Fig. 2(b)]. Note that the perturbation velocities represent fluctuations about the horizontally averaged flow profile. These fluctuations include wave-like motions as well as turbulence. To aid interpretation, we divide the life cycle into three stages. From \( t = 0 \) to \( t = 1200 \) s, the flow consists of the original parallel shear flow, plus a growing, two-dimensional KH wave train. From \( t = 1200 \) to \( t = 1800 \) s, three-dimensional secondary instability grows until its energy reaches a level somewhat below that of the streamwise and vertical flow components. (The growth of the secondary instability actually begins before \( t = 1200 \) s. The boundaries of this interval are defined based on geometrical properties of the velocity gradients to be discussed in Sec. V.) The growth of the three-dimensional secondary instability coincides with a dramatic increase in turbulence intensity, as quantified by \( R_s \), \( R_b \), and Cox. This is the period we identify with the onset of turbulence. After the billows reach maximum amplitude, the KH waves exhibit a quasiperiodic exchange of energy between streamwise and vertical motions which also involves the potential energy reservoir (not shown). This oscillation corresponds to the nutation of the elliptical vortex core.\(^{34}\)

Turbulence develops largely during the phases of the oscillation in which potential energy is being converted to kinetic energy, a process very reminiscent of the breaking of a surface wave. Early in the turbulent phase, vertical kinetic energy decreases rapidly until it reaches a level very similar to that of the spanwise component (near \( t = 3600 \) s). Beyond this point, the oscillations cease and the disturbance decays monotonically. The simulation was continued until the disturbance energy was smaller than the energy of the noise field with which the model was initialized. Energy conversions that occur during the KH life cycle will be discussed in detail in a separate paper.

The relationship of our simulated flows to the classical picture of fully developed turbulence is evident in the evolving probability distribution of the local turbulent kinetic energy dissipation rate, \( \epsilon \) (Fig. 3). At \( t = 0 \), the initial shear layer plus perturbation appears as a smooth distribution of \( \epsilon \). The maximum near \( \epsilon = 10^{-9} \) W/kg represents the parallel shear layer. By \( t = 565 \) s, the distribution has taken on a bimodal form complicated by numerous small peaks. Rapid initial spreading of the shear layer has decreased the corresponding local maximum from \( 10^{-9} \) to \( 5 \times 10^{-10} \) W/kg. The
broad distribution of smaller \( \epsilon \) values has been replaced by a jagged peak near \( \epsilon = 10^{-11} \) W/kg as the effects of weak flow distortions from the initial perturbation diffuse through the domain. The jaggedness evident during the rollup phase (\( t = 565 \) s, \( t = 1131 \) s) reflects the coalescence of regions of quasistreamwise \( \epsilon \) in the cores and braids of the merging KH vortices. Between \( t = 1131 \) s and \( t = 2262 \) s, three-dimensional secondary instabilities trigger the transition to turbulence.

By \( t = 2262 \) s, the distribution is dominated by a single, smooth peak centered near \( \epsilon = 3 \times 10^{-9} \) W/kg. The shape of this peak approximates the lognormal form, predicted for fully developed turbulence by Kolmogorov.\(^{35} \) (The history of the lognormal distribution and its modern variants is reviewed by Frisch.\(^{3} \)) The jagged distribution of lower \( \epsilon \) values represents the laminar flow surrounding the turbulent layer. For the case \( t = 3111 \) s, we also include the distribution function pertaining only to the averaging volume, \( V_s \), to demonstrate that \( V_s \) effectively isolates the most-turbulent region of the flow. The skewness of \( \log_{10}(\epsilon) \) within \( V_s \) is \( -0.25 \), indicating that the departure from lognormality is small. A slight, negative skewness was also observed by Wang et al.\(^{36} \) in simulations of stationary, homogeneous, isotropic turbulence with Reynolds number larger than we achieve here. Wang et al.’s parameter \( Q \), which is unity for a perfectly lognormal distribution, is equal to 0.997 within \( V_s \). As the simulation is continued, turbulence decays, and the peak shifts gradually to lower values. In the long-time limit, the distribution loses its lognormal character as turbulence is replaced by laminar gravity waves and \( \epsilon \) is dominated by the mean shear, which is by this time nearly uniform in space.

We have seen that the probability distribution function for \( \epsilon \) exhibits a dramatic change between the early and late phases of the simulation. Early on, when the flow is dominated by KH billows and associated transitional flow structures, the distribution is jagged. Later, the distribution exhibits a smooth, nearly lognormal peak indicative of the random, multiplicative character of a turbulent cascade.\(^{37} \) We conclude that, although the Reynolds number of this simulation is decidedly finite, the dynamics of the dissipation range exhibit important characteristics in common with fully developed turbulence. In Sec. V, we will conduct detailed comparisons of dissipation range statistics from our simulations with those expected in fully developed turbulence, in order to assess the applicability of the latter idealization to naturally occurring flows. As preparation for this, we devote Sec. IV to a discussion of the manifestly anisotropic structure of the large scales.

IV. ENERGY-CONTAINING SCALES

In this section, we describe the geometry of the largest eddies in terms of the velocity anisotropy tensor \( b \) and its invariants (17) and (18). We begin by examining \( b \) as a function of the vertical coordinate for a sample flow field. We display \( b \) computed as a depth-dependent tensor, i.e., by interpreting the angle brackets in (17) as representing horizontal averages (Fig. 4).

For reference, Fig. 4(a) shows the mean temperature and velocity profiles. The velocity fields tend to be highly anisotropic near the boundaries. This anisotropy is prolate in form [\( \Pi_0 > 0 \), as shown in Fig. 4(d)] due to the dominance of the streamwise component of the velocity fluctuations. Near the center of the domain, the degree of anisotropy is considerably reduced. The vertical velocity becomes similar in magnitude to the streamwise velocity, while the spanwise component remains smaller than the other two [Fig. 4(b)]. The strong negative correlation between the streamwise and vertical velocity fluctuations [Fig. 4(c)] indicates that these fluctuations are configured so as to draw energy from the background shear. The second invariant of \( b \) remains significantly nonzero even at the center of the domain. To put the observed degree of anisotropy in perspective, we note that, in laboratory studies of the return to isotropy in initially anisotropic turbulence,\(^{29,38} \) a typical starting value of \( \Pi_0 \) is 0.05. That value is typical of the closest approach to isotropy in our simulations. The third invariant oscillates between positive and negative values, reflecting fluctuations in the relative magnitudes of the streamwise and vertical velocity components [Fig. 4(b)].

We next examine the time-dependent behavior of \( b \) over an entire simulation (Fig. 5). (Hereafter, \( b \) is defined using averages over \( V_s \).) Near \( t = 0 \), \( b_{11} \) is the largest of the diagonal components, \( b_{33} \) is somewhat smaller, and \( b_{22} \) is close to \( -\frac{1}{2} \). This indicates that the flow is two dimensional. The streamwise velocity dominates, while the spanwise velocity is close to zero. During this time, \( b_{13} \) is negative, indicating the presence of a significant Reynolds stress working against
the mean shear to transfer energy into the growing disturbance. During the 3d growth and transition phases, $b_{11}$ and $b_{33}$ oscillate, as do the kinetic energy components to which they correspond [cf. Fig. 2(a)]. During this time, spanwise kinetic energy (expressed here through $b_{23}$) grows in response to secondary instability, although it does not represent a significant fraction of the disturbance kinetic energy ($b_{23} > -1/3$) until the 3d growth phase is complete and the transition to turbulence is underway. During 3d growth and transition, $b_{11}$ also oscillates, revealing a quasiperiodic exchange of energy between the KH vortex and the mean flow. In the initial growth phase, $b_{13}$ reaches values near $-0.30$. During the decay phase, as was seen in Fig. 2, the spanwise and vertical velocities are very similar in mean magnitude. Transition leaves the value of $b_{13}$ near $-0.10$, and that value subsequently decays to zero as the turbulence ceases to draw energy from the mean flow. Values of $b_{13}$ found here are comparable with the value $-0.13$ that is typical in near-wall turbulence.

A striking aspect of this anisotropy analysis is the degree to which the invariants, $II_b$ and $III_b$, adhere to the limiting values associated with two-dimensional and axisymmetric flow [Figs. 5(c) and 5(d)]. During the 2d and 3d growth phases, $II_b$ and $III_b$ have values characteristic of two-dimensional flow, with $II_b$ near its upper limit and $III_b$ near its lower limit. Early in the transition phase, the invariants depart from their two-dimensional limits, moving to a state of oblate axisymmetry in which the spanwise velocity tends to be smaller than the other two velocity components. The velocity field makes its closest approach to isotropy (as measured by $II_b$), at the end of the transition phase. By this point, the invariants have settled into a configuration characteristic of prolate axisymmetry (with streamwise velocity dominant), where they remain throughout the decay phase.

Figure 6 shows values of $II_b$ and $III_b$ for all eight simulations. In general, the progression in time is counterclockwise on the $III_b$--$II_b$ plane. The energy-containing scales remain distinctly anisotropic in all cases. Also, the tendencies toward oblate axisymmetry in the transition phase and prolate axisymmetry in the decay phase are present in all simulations.

We have examined the form of the asymmetry of the large scales and how that asymmetry controls the flux of energy from the mean shear to the turbulence. In the remainder of the paper, we assess the degree to which this anisotropy is reflected in the structure of the smaller scales of motion.

V. VELOCITY GRADIENTS

We turn now to an examination of the structure of eddies in the dissipation range. Anisotropy in these small-scale mo-
tions is revealed through the vorticity and dissipation anisotropy tensors and through streamwise cospectra and correlation-coefficient spectra of the velocity field. We then assess the accuracy of some popular approximations to $\langle \epsilon \rangle$ based on assumptions regarding isotropy in the dissipation range.

A. Tensor measures of anisotropy

Figure 7 shows the evolution of the vorticity anisotropy tensor (19) during a typical simulation. Throughout the simulation, the vorticity field exhibits significant anisotropy. Although the mean-square vorticity components in this reference frame approach a common value between $t=2400$ and $t=4200$ s, the off-diagonal element $v_{13}$ is positive during this interval. As a result, the second invariant remains significantly nonzero. It is, however, closer to zero during the most-turbulent interval of the simulation than is $I_{13}$. (This comparison is not entirely fair, because $v$ contains the mean profile while $b$ does not. If mean profiles were retained in the calculation of $b$, the difference between $I_{13}$ and $I_{10}$ would be greater.) The positive value of $v_{13}$ suggests a preference for vortices aligned midway between the streamwise and vertical directions, i.e., in the direction of the principal axis of extensional strain for the mean shear.

We now look more closely at the form of the anisotropy. Time-averaged values of the elements in the vorticity anisotropy tensor taken shortly after transition ($t=2400–3600$ s) are

$\bar{v} = \begin{bmatrix} 0.023 & 0.001 & 0.104 \\ 0.001 & 0.001 & 0.001 \\ 0.104 & 0.001 & -0.025 \end{bmatrix}$

(Figure 7 shows that $v$ is statistically stationary during this averaging interval.) With its dominant components residing in the $(1,3)$ and $(3,1)$ positions, this tensor is nearly proportional to the mean strain tensor:

$\bar{S} = \begin{bmatrix} 0 & 0 & \frac{1}{2}u_{xz} \\ 0 & 0 & 0 \\ \frac{1}{2}u_{xz} & 0 & 0 \end{bmatrix}$

The eigenstructures of the two tensors are thus very similar. There are three eigenvalues which sum to zero: one positive, one near zero, and one negative. The first eigenvector (i.e., that corresponding to the positive eigenvalue) lies in the $x-z$ plane at a 45° angle above the $x$ axis. The third eigenvector, corresponding to the negative eigenvalue, is also in the $x-z$ plane, directed 45° below the $x$ axis. The middle eigenvector is parallel to the $y$ axis. In the coordinate system defined by these eigenvectors, the vorticity field appears as three orthogonal components: the principal components. The strongest of the three is aligned with the extensional strain, and suggests the tilted vortices that are ubiquitous in sheared turbulence. The weakest component is aligned with the compressive strain. The middle component represents the spanwise vorticity that dominates the flow in its early and late stages.

A crucial property of the principal component vorticities is that the mean square of the middle component is nearly equal to the arithmetic average of the other two components (and thus $I_{13}$ is near zero). It is not clear why this should be so. Because of this property, this manifestly anisotropic flow exhibits an important property in common with isotropic flow: In the original reference frame (oriented with the mean flow), the mean squares of the three components of the vorticity field are nearly equal [Fig. 7(a)]. In other words, knowledge of one Cartesian component of the mean square vorticity allows one to deduce the other two components just as if the vorticity field were isotropic.

Typical evolution of the invariants of $v$ for the transition and decay phases of all eight simulations is shown in Figs. 8(a) and 8(c). To facilitate comparison between runs with different inherent time scales, we now measure time in buoyancy periods, i.e., $t' = f_0^t N(t') \, dt' / 2\pi$. In most cases, the pattern is similar to that seen in Fig. 7. $I_{13}$ and $I_{10}$ remain close to their one-component values (2/3 and 2/9, respectively) throughout the initial period of two-dimensional
growth. The invariants then decrease rapidly during the transition to turbulence, which occurs after 2–3 buoyancy periods. (The time to transition may be a strong function of initial conditions.) The subsequent phase of minimal anisotropy, during which $\Pi_{\nu}^{1/2}$ is in the range 0.1–0.2, lasts for another 2–3 buoyancy periods. The invariants then increase back to their one-component values as the flow gradually relaxes to the stable, two-dimensional end state. The only exception to this pattern occurs in the simulation R03P1, shown with diamonds in Figs. 8(a) and 8(c). This was the least turbulent of our simulations, in the sense that $R_b$ was too large (0.16) and $Re_0$ too small (~2000) for vigorous turbulence to develop.

The dependence of local isotropy upon turbulence intensity is illustrated more vividly in Figs. 8(b) and 8(d), which show the vorticity invariants as functions of the buoyancy Reynolds number. Time is indicated by symbol shading; time progression is clockwise, from the lightest shade to the darkest. After the transition to turbulence is complete and $R_b$ has attained its maximum value (the lower right-hand corner of the frame), the evolution of the invariants is tied very closely to the evolution of $R_b$. In other words, the invariants “forget” the effects of initial conditions (and the Prandtl number of the fluid) and subsequently depend only on $R_b$. This result strongly supports the notion that isotropy of the dissipation range is controlled primarily by the spectral separation between the smallest and largest scales of motion.

The results shown in Fig. 8(b) permit us to guess at what might occur at Reynolds numbers higher than those attained in the present simulations. $\Pi_{\nu}$ cannot approach zero (within the scatter of the data) for $R_b$ much smaller than $O(10^5)$, unless the form of the dependence changes qualitatively for $R_b > O(10^5)$. On the other hand, it is easy to imagine that $\Pi_{\nu}$ could asymptote to some small but nonzero value as $R_b \rightarrow \infty$.

We now return to a consideration of the nature of the anisotropy exhibited by the vorticity field after the transition to turbulence. In the slightly idealized picture described in connection with Fig. 7, the invariant $\Pi_{\nu}$ is proportional to $v_{13}$, while $\Pi_{\ell}$ is zero. In fact, the middle eigenvalue of $\nu$ is not quite zero, but rather fluctuates about that value. This causes $\Pi_{\ell}$ to fluctuate about zero in a similar manner. Figure 8(d) reveals a common tendency for $\Pi_{\ell}$ to become negative near the end of the turbulent regime, just before the flow starts to relax towards the two-dimensional configuration. The sign change is associated with values of $R_b$ close to 10.

This intriguing regularity in $\Pi_{\ell}$ can be understood in a qualitative sense via closer examination of Fig. 7(a). Shortly after transition, when $R_b$ is large, flows tend to develop positive $\Pi_{\ell}$ as the middle principal component of vorticity (which is generally close to the spanwise direction) becomes smaller than its isotropic value. During this phase, the vorticity field is slightly prolate, being dominated by its streamwise ($x$) component. At later times, streamwise and vertical vorticity decay, so that spanwise vorticity increases in relative magnitude as the flow relaxes toward the two-dimensional end state. The latter state is also prolate, since the vorticity is dominated by its spanwise component. Between these two phases, when the turbulence is just beginning to decay, there is inevitably an intermediate state in which streamwise and spanwise vorticity are similar in magnitude, while vertical vorticity is considerably smaller. That intermediate state is oblate, and is therefore characterized by $\Pi_{\ell} < 0$.

The invariants $\Pi_d$ and $\Pi_{13}$ behave similarly to their counterparts $\Pi_{\ell}$ and $\Pi_{13}$ in many respects (Fig. 9). Both invariants are large, indicating the dominance of a single eigenvector of the anisotropy tensor, at early and late times. The smallest values of $\Pi_d$ are attained in the interval $3 < \tau_N < 4$, and are somewhat smaller than the smallest values...
of II_v. The third invariant, III_d, displays very little tendency to take negative values during the interval of minimal anisotropy. Like the vorticity invariants, II_d and III_d are accurately predicted by the buoyancy Reynolds number once the transition to turbulence is complete [Figs. 9(b) and 9(d)].

We turn now to an examination of the dependence of the dissipation anisotropy invariant II_d on parameters expressing the relative strength of the large-scale shear. Corrsin27 and others have suggested that the dissipation subrange should be effectively insulated from anisotropy due to the large-scale shear if \( R_s \) is sufficiently large. In contrast, Durbin and Speziale7 have argued that the dissipation scales must remain anisotropic regardless of \( R_s \), except possibly in the limit that the shear number, \( S^* \), goes to zero. In the present results, \( R_s \) evolves in a manner similar to \( R_b \), growing to a maximum then decaying [cf. Fig. 2(b)], while \( S^* \) decreases from starting values of order 10^2 to asymptotic values not far from the value 6 found previously in both laboratory and numerical experiments.7 \( R_s \) proves to be an excellent predictor of dissipation anisotropy (as quantified by II_d) in the present experiments [Fig. 10(a)], whereas II_d shows very little correlation with \( S^* \) [Fig. 10(b)].

The present results are not necessarily inconsistent with the predictions of Durbin and Speziale. Even the smallest values of II_d achieved here indicate nonzero anisotropy. Suppose that the present simulations could be performed on a computer of arbitrary size, so that the limit \( R_s \to \infty \) could be approached. It seems likely that \( S^* \) would remain of order unity or larger in such simulations [Fig. 10(b)]. (In fact, in a state of production-dissipation balance, \( S^* \) must remain larger than unity.)7 The results of Durbin and Speziale therefore predict that the small scales would remain anisotropic as \( R_s \to \infty \). Although Fig. 10(a) suggests otherwise, it does not rule out the possibility that II_d would asymptote to some small, but nonzero value. Another possibility suggested by the present results is that II_d asymptotes to zero in the high Reynolds number limit, but II_v remains nonzero [Fig. 8(b)]. This would indicate anisotropy in the vorticity field that is not reflected in the dissipation tensor.
B. Spectral measures of anisotropy

We now examine anisotropy over the entire range of length scales as revealed by streamwise Fourier spectra of the velocity field (23) and (24). Figure 11(a) shows streamwise spectra of kinetic energy production and (approximate) dissipation for a strongly turbulent flow. This example was taken from simulation R16P1 at \( t = 3111 \text{s} \), shortly after transition. It is clear that production and dissipation are controlled by motions in different spectral ranges. Production occurs mainly in the spectral range \( k \approx k_C \), where eddies are strongly deformed by the mean shear. (Wave numbers are scaled by the Kolmogorov wavenumber, \( k_K = 1/L_K \).) The dissipation range begins at about \( 0.1k_K \), as is typical in turbulent flow.\(^{39}\) The Ozmidov wave number, below which eddies are strongly damped by stratification, is located near the small-scale end of the production range. Regions of negative production indicate that the large scales are still substantially wavelike.

In Fig. 11(b), we display the component energy spectra \( E_{11} \), \( E_{22} \), and \( E_{33} \), scaled by \( k^{5/3} \) so as to test for the presence of an inertial subrange. At low wave numbers, the three curves differ significantly; energy associated with spanwise motions (dashed curve) is by far the smallest. The three curves converge near \( k = 0.05k_K \) and remain nearly horizontal until \( k \approx 0.20k_K \), a range of about one half decade. Within this range, the spectral level is consistent with a value 1.5 for the Kolmogorov constant. The ratios of the transverse spectra to the longitudinal spectrum [Fig. 11(c)] compare well with the theoretical value of 4/3 in the inertial range (cf. Fig. 10 of Gargett et al.\(^{16}\)). At higher wave numbers, the component spectra roll off [Fig. 11(b)] while remaining nearly equal. In summary, Fig. 11 indicates a highly anisotropic production range at large scales, a nearly isotropic dissipation range at small scales, and an intervening inertial range extending for about one half decade.

The flow shown in Fig. 11 represents one of the most strongly turbulent states achieved in these simulations, with \( R_b = 100 \). We turn next to an examination of a more typical case (Fig. 12), taken from the same simulation (R16P1) long after transition (\( t = 6222 \text{s} \)). By this time, \( R_b \) has decreased to 25. This flow exhibits distinct production and dissipation ranges [Fig. 12(a)], though they are less well separated than in the previous case. The component spectra are nearly equal in the dissipation range, which begins near \( k = 0.1k_K \) as before. Near this wave number, the spectral slope is close to \(-5/3\), the amplitude is consistent with \( C_K = 1.5 \), and the transverse/longitudinal ratios [Fig. 12(c)] are not far different from 4/3. However, this inertial subrange-like behavior does not extend over any significant region of the spectrum, but rather appears only tangentially at the beginning of the dissipation range. Thus, the lower \( R_b \) case shown in Fig. 12 exhibits a mildly anisotropic production range and a nearly isotropic dissipation range, separated by the merest hint of an inertial range.

A more stringent criterion for isotropy than \( E_{11} = E_{22} = E_{33} \) is the vanishing of the Reynolds stress spectra \( E_{12} \), \( E_{23} \), and \( E_{13} \). In our simulations, the first two of these vanish in the dissipation range as expected. \( E_{13} \), however, does not vanish but rather changes sign; over most of the dissipation range, we find a small positive Reynolds stress. This sign change can be seen from the production spectrum in Fig. 12(a), which is proportional to \( E_{13} \). The sign of the Reynolds stress in the dissipation range is shown more clearly by the correlation-coefficient spectrum \( C_{13} \) [Fig. 12(c)].

This correlation between streamwise and vertical velocity fluctuations in the dissipation range suggests that the straining effects of the large-scale shear are felt even at the smallest scales of motion. Similar sign reversals have been noted previously in results from both numerical simulations\(^{44,45}\) and laboratory experiments.\(^{6,44,46–48}\) These experiments have covered a range of \( R_s \) extending from less than 100 to nearly 3000. Comparison between these various studies is complicated by differences in the form of the spectra used (\( E_{13} \), \( k_1E_{13} \), and \( C_{13} \)), in order of increasing sensitivity to small scales), but it seems that the effect is less pronounced at higher \( R_s \). For example, the DNS experi-
ments of Holt et al. employed $R_s$ of order 100 and less. In those results, the sign reversal is visible in the $E_{13}$ spectrum, even though the amplitude of that spectrum is characteristically small at high wave numbers. The sign reversal is more visible in the results of Antonia et al. (derived from DNS and small-scale laboratory experiments) and Shaﬁ and Antonia (from laboratory experiments), who used similarly small $R_s$, but displayed the compensated spectrum $k_1E_{13}$. Much higher values of $R_s$ have been reached in large-scale wind tunnel experiments. In these cases, the sign reversal can only be discerned in the highly sensitive $C_{13}$ spectrum, and is often obscured by underresolution of the dissipation range. The earliest such results were presented by Mestayer, who achieved $R_s \sim 2800$ in a wind tunnel. The sign reversal in his results is slight, but visible in the $C_{13}$ spectrum (his Fig. 13(a)]. Similarly, wind tunnel results of Saddoughi and Veeravalli, with $R_s \sim 1000–2500$, show a hint of a reversal in $C_{13}$ [their Figs. 20(b) and 20(c)]. Antonia and Raupach observed a sign reversal in $C_{13}$ in a wind tunnel with $R_s$ ranging from 300 to 1500. In each of these cases, the sign reversal occurred at $k_1/k_K \sim 0.1$, the beginning of the dissipation subrange. Saddoughi and Veeravalli also investigated cases with much higher $R_s$ [their Figs. 20(a) and 20(c)] in which there is no evidence of sign reversal, but this result is inconclusive since the dissipation range was not well resolved. Thus, the evidence to date indicates that sheared turbulence with $R_s$ less than a few thousand exhibits positive correlation between streamwise and cross-stream velocity components in the dissipation range.

The results shown in Fig. 12 are entirely consistent with those quoted earlier. Our values of $R_s$ are toward the low end of the range of previous investigations, and the sign change is visible even in the relatively insensitive $k_1E_{13}$ form of the spectrum [Fig. 12(a)]. In the $C_{13}$ spectrum [Fig. 12(c)], the positive correlation at small scales is more obvious. As in the previous investigations, the sign reversal occurs at the beginning of the dissipation range.

We now test the generality of the above-suggested conclusions by examining results from several turbulent flow solutions (Fig. 13). The zero crossing, at which point the cospectrum becomes positive, appears to be determined by the Kolmogorov scale. Speciﬁcally, it occurs at $k_1/k_K \sim 0.1–0.2$, the beginning of the dissipation range. Spectra plotted against $k_1/k_C$, where $k_C = (S^3/\varepsilon)^{1/2} = 1/L_C$ show no such consistency in the location of the zero crossing [compare Figs. 13(a) and 13(b)]. The fact that the wave number of the zero crossing scales with $k_K$ and not with $k_C$ suggests that the anomalous correlation is a property of the dissipation subrange, and its spectral extent is independent of straining by large-scale motions. In contrast, the amplitude of the anomalous correlation is clearly a function of $R_s$ (shown in Fig. 13 by symbol size). When $R_s$ is large, i.e., there is a wide spectral separation between the large scales and the dissipation range, the correlation is small. At the largest $R_s$ attained here, $C_{13}$ peaked near 0.15. (In the much larger $R_s$ attained in the wind tunnel experiments cited above, $C_{13}$ tended to peak at values near 0.05, or smaller.) As the turbulence decays, however, the effect of the largest eddies on the dissipation range increases, and the anomalous correlation also increases. In the late stages of turbulence decay, the peak in the correlation-coefﬁcient spectrum approaches unity.

This agreement between results from experiments made using a wide range of numerical and laboratory techniques suggests that the correlation between streamwise and vertical velocities at large wave numbers is neither a numerical effect nor a measurement error. The correlation appears to be controlled by a combination of viscous and large-scale straining effects. The degree of correlation is determined by the effect of large-scale straining on the dissipation scales, whereas the spectral range over which the correlation occurs is independent of the large-scale straining, and coincides closely with the dissipation range.

Toward a physical interpretation of the anomalous correlation, we note that the deformation of spanwise vortices by the large-scale shear is expected to lead to expansion in
the direction of the principal axis of extensional strain (and compression in the perpendicular direction). The resulting deformed eddies will exhibit a positive correlation between \( u \) and \( w \). At large scales, this effect competes with the tendency for eddies with the opposite tilt to be amplified by shear. Due to the finite bandwidth of that instability, the mean shear cannot drive unstable vortices on the smallest scales. (Shear associated with larger vortices leads to instability.) Another competing influence is the vortices that arise due to vortex stretching along the extensional strain of the mean flow. Such vortices are characterized by negative correlation between \( u \) and \( w \). In conclusion, it is not difficult to see why motions with positive \( C_{13} \) exist, but it is also easy to identify mechanisms which favor the opposite correlation and which could dominate some areas of the spectrum.

C. Estimates of the kinetic energy dissipation rate

The kinetic energy dissipation rate is often estimated in terms of just one of its terms, i.e.,

\[
\langle \epsilon \rangle \approx C_{ij} \langle u_{i,j}^2 \rangle
\]

(no summation on \( i, j \)), in which the constant \( C_{ij} \) is equal to 15 if \( i = j \), 15/2 otherwise. These approximations become exact in isotropic flow, and are expected to be accurate if the dissipation subrange is isotropic. Figure 14 shows each of these approximations, applied to our data and normalized by the true value of \( \langle \epsilon \rangle \). All nine of these approximations are accurate to within 10%–20% at the highest values of \( R_b \) attained here (near 100). At lower \( R_b \), however, such approximations may err by an order of magnitude or more.

The poorest approximations are those that involve streamwise derivatives [Figs. 14(a), 14(d), and 14(g)]. The background shear tends to elongate eddies in the streamwise direction, so that the corresponding derivatives are relatively small. The best approximations are based on \( \partial u / \partial y \) and \( \partial v / \partial z \) [Figs. 14(b) and 14(f)]. The only approximation that consistently overestimates \( \langle \epsilon \rangle \) is that based on the vertical shear of the streamwise velocity [Fig. 14(c)]. Although the background shear has been removed, this corresponding perturbation shear is significantly larger than the other velocity gradients. Errors in \( \langle \epsilon \rangle \) due to anisotropy in the dissipation range are particularly dangerous because they lead to errors in \( R_b \) (since the latter is itself proportional to the dissipation rate). In other words, the quantity we use to assess the potential for anisotropy is only estimated accurately when the turbulence is isotropic! The data analyst may thus be misled as to the degree of anisotropy to be expected in the flow.

Comparisons between the relative contributions of the various terms to \( \langle \epsilon \rangle \) are in excellent qualitative agreement with the results of Itsweire et al.\(^{17} \) (cf. their Table I, cases Ri=0.075 and Ri=0.21). The latter were obtained from DNS of turbulence in a uniformly sheared and stratified environment. The present results also agree well with those derived by Gargett et al.\(^{16} \) from spectra of turbulence occurring in a fjord. On the basis of those data, Gargett et al. concluded that \( \langle \epsilon \rangle \) can be reliably estimated from a single term of the form (26) when \( R_b > 200 \). While that value of \( R_b \) is not attained in the present simulations, extrapolation of the results shown in Fig. 14 suggests that all of the isotropic formulas will be accurate to within about 10% above that value. In practice, many of the isotropic approximations evaluated here are accurate enough that errors due to anisotropy are likely to be smaller than other sources of uncertainty for \( R_b \) as small as \( O(1) \).

We next examine the ratios \( K_1, K_2, K_3, \) and \( K_4 \) (22), which have proved useful in the characterization of anisotropy in laboratory data.\(^{31} \) In isotropic flow, all of these ratios should be unity. An alternative idealization for shear flow is the the state of axisymmetry about the streamwise direction. Flows exhibiting that symmetry have \( K_1 = K_2 \) and \( K_3 = K_4 \). The results displayed in Fig. 15 suggest that axisymmetry may be a useful approximation for these flows at low \( R_b \).

In Fig. 16, we test the accuracy of four approximations to \( \langle \epsilon \rangle \), each of which is exact in axisymmetric flow:}

\[
\langle \epsilon_1 \rangle = \nu \left[ \frac{2}{3} \langle u_x^2 \rangle + 2 \langle u_y^2 \rangle + 2 \langle w_z^2 \rangle + \frac{8}{3} \langle w_x^2 \rangle \right],
\]

\[
\langle \epsilon_2 \rangle = \nu \left[ -\langle u_x^2 \rangle + 2 \langle u_y^2 \rangle + 2 \langle w_z^2 \rangle + 8 \langle w_x^2 \rangle \right],
\]

\[
\langle \epsilon_3 \rangle = \nu \left[ \frac{2}{3} \langle u_x^2 \rangle + 2 \langle u_y^2 \rangle + 2 \langle v_x^2 \rangle + \frac{8}{3} \langle v_z^2 \rangle \right],
\]
FIG. 14. Contributions to the kinetic energy dissipation rate, \( \langle \varepsilon \rangle \), from each of the mean squared velocity derivatives, averaged over \( V_s \) and expressed as a fraction of the true value of \( \langle \varepsilon \rangle \). Each ratio is unity in isotropic flow. Symbol sizes indicate the Prandtl number, with \( Pr > 1 \) represented by the smaller symbols. The time is indicated by shading, as shown on the bar above Fig. 9(a). For clarity, data from \( t < 1800 \text{ s} \) (i.e., preturbulent cases) are excluded.

FIG. 15. Isotropy ratios as defined in (22). Each ratio is unity in isotropic flow. Symbol sizes indicate the Prandtl number, with \( Pr > 1 \) represented by the smaller symbols. The time is indicated by shading, as shown on the bar above Fig. 9(a).
\[ \langle \epsilon_4 \rangle = \mathcal{E} \left[ -\langle u_{x}^{2} \rangle + 2\langle u_{y}^{2} \rangle + 2\langle u_{z}^{2} \rangle + 8\langle \omega_{y}^{2} \rangle \right] \]  
(30)

All formulas are comparable in accuracy to the best of the isotropic approximations [e.g., Figs. 14(b), 14(e), 14(f), and 14(i)].

These results compare well with similar analyses performed on data from a laboratory experiment on duct flow by Antonia et al.\textsuperscript{30} (cf. their Figs. 9 and 12). In that case, departures from isotropy occurred not as the turbulence decayed in time (as in the present case) but rather as the measurement location approached the wall. \( K_1 \) and \( K_2 \) were greater than unity while \( K_3 \) and \( K_4 \) were less than unity. George and Hussein\textsuperscript{31} list numerous other experiments in which similar results have been obtained. As in the present case, Antonia et al.\textsuperscript{30} found that \( \epsilon_1 \) underestimates \( \langle \epsilon \rangle \) and \( \epsilon_2 \) overestimates \( \langle \epsilon \rangle \) as the flow departs from isotropy. Antonia et al.\textsuperscript{30} also computed the isotropic approximation based on \( \partial u / \partial x \) and found it to be a considerably poorer approximation to \( \langle \epsilon \rangle \) than the axisymmetric approximations. Our own results agree with this finding [Fig. 14(a)], but show in addition that other formulas based on isotropy do just as well as the axisymmetric approximations.

VI. SCALAR GRADIENTS

We complete our analyses with an investigation of anisotropy in the scalar gradient field. In the most-turbulent regimes, the scalar gradient is isotropic except for a negative correlation between the streamwise and vertical components (Fig. 17). This indicates a preference for gradients aligned with the principal compressional strain of the mean shear. As in the case of the vorticity field, the orientation and strength of the anisotropic structures is such that the root mean square components of the temperature gradient vector remain nearly equal in a reference frame aligned with the mean flow. In other words, the mean squared gradient in the spanwise direction is close to the average of the mean squared gradients in the orthogonal directions defined by the compressional and extensional eigenvectors of the mean strain tensor. There is, however, an important difference between the temperature gradient and vorticity fields in this respect: The spanwise vorticity component is associated with the two-dimensional flow that dominates in weakly turbulent regimes, whereas the spanwise component of the temperature gradient is small in those regimes, and is instead associated with fully three-dimensional flow. Note also that the phase in which anisotropy is smallest is very brief in comparison with results found for the velocity gradients.

The variance of the scalar gradient is relatively small at the center of the turbulent layer [Fig. 18(a)]. This reflects the
The fact that scalar mixing at the center of a stratified shear layer tends to proceed more rapidly than momentum mixing.\(^2,49\)

As has been found in previous investigations,\(^50–52\) the skewness of the streamwise temperature gradient is near \(-2\), while that of the vertical temperature gradient is near \(-1\). Not surprisingly, this skewness reveals a significant dependence on Prandtl number, with high-Pr cases tending to be more isotropic than low-Pr cases for a given \(R_b\). This is because high-Pr flows feature enhanced small-scale temperature variance, which is relatively immune to the aligning effects of gravity. This Prandtl number dependence is removed when one plots the invariants against the Cox number [Figs. 19(c) and 19(f)], but the overall degree of scatter is only slightly reduced (and preturbulent cases now lie far from the curve suggested by the turbulent cases).

We now test isotropic approximations to the mean squared scalar gradient, to which the scalar variance dissipation rate is proportional via \(\chi = 2 \kappa | \nabla \theta' |^2 \). In isotropic flow, the three Cartesian components of \(| \nabla \theta' |^2\) are equal, so that

\[
3 \langle \theta'^2 \rangle = 3 \langle \theta_x'^2 \rangle = 3 \langle \theta_z'^2 \rangle = | \nabla \theta' |^2.
\]

Therefore, measurement of a single component suffices to determine the mean squared gradient (and hence \(\chi\)).

In Fig. 20, we display averages over \(V_s\) of the three isotropic approximations, expressed as ratios of the true value of the mean squared gradient. At the highest Cox numbers attained in these simulations, the isotropic approximations are accurate to within a few tens of percent. Note that these isotropic estimates are unaffected by the strong correlation between \(\partial \theta' / \partial x\) and \(\partial \theta' / \partial z\) shown in Fig. 17. In weaker turbulence, the effects of departures from isotropy are clearly evident. Horizontal gradients are relatively small, while vertical gradients are relatively large, even though the mean vertical gradient is subtracted out. From these results, we conclude that isotropic estimates of the mean squared scalar gradient (and thus of \(\chi\)) based on a single component

FIG. 18. Profiles of statistical moments of the temperature gradient components, taken over horizontal planes, for run R10P1, \(t = 4242\) s. (a) Standard deviations, normalized by the initial scalar fluctuation and length scales. (b) Skewnesses. In both frames, solid, dashed, and dotted curves correspond to \(\theta_x, \theta_y,\) and \(\theta_z,\) respectively.

FIG. 19. Invariants of the scalar gradient anisotropy tensor for all runs. Abcissae are the time in buoyancy periods (a), (d), the buoyancy Reynolds number (b), (e), and the Cox number (c), (f). In (b), (c), (e), and (f), symbol sizes indicate the Prandtl number, with \(\text{Pr} > 1\) represented by the smaller symbols. The time is indicated by shading, as shown on the bar above (a).
are accurate to within a few tens of percent when Cox > O(10^2), but may be highly misleading at smaller Cox. Estimates based on spanwise gradients seem to be the most robust in this respect. Laboratory experiments by Thoroddsen and Van Atta on inhomogeneous grid turbulence reveal similar results; streamwise scalar gradients tend to be considerably smaller than spanwise and cross-stream gradients.

VII. SUMMARY

Turbulence in a stratified shear layer at moderate Reynolds number exhibits significant anisotropy on all scales, due to the straining action of the mean shear and the suppression of vertical motions by buoyancy forces. The flow is strongly anisotropic during the preturbulent phase of the flow evolution. Anisotropy decreases dramatically with the transition to turbulence, then increases again as the turbulence decays and the flow relaxes to a stable, parallel configuration.

Large-scale anisotropy of the velocity field is such as to generate Reynolds stresses which extract energy from the background shear flow. During the turbulent phase, the vorticity field exhibits anisotropy in the form of a correlation between streamwise and vertical vorticity components. This anisotropy leaves the three Cartesian components of the vorticity (in a reference frame aligned with the mean flow) nearly equal in magnitude, so that an estimate of enstrophy based on knowledge of a single component and the (incorrect) assumption of isotropy would give a valid result.

After the onset of turbulence, anisotropy of the velocity gradients is controlled entirely by the Reynolds number. This dependence is illustrated by the tight collapse of anisotropy parameters when plotted against $R_s$ and $R_b$ [Figs. 8(b), 8(d), 9(b), 9(d), and 10(a)]. This result is one of the central outcomes of the present study, and strongly supports the idea that small-scale anisotropy depends on the spectral separation between the largest and smallest scales. Extrapolation from presently accessible Reynolds numbers suggests that the vorticity field may become isotropic at sufficiently large values $[R_s \sim O(10^3)]$. However, we cannot rule out the possibility that some anisotropy persists even in the high Reynolds number limit.

Scaled energy spectra (Figs. 11 and 12) indicate anisotropy in the production range, but are consistent with isotropy in the dissipation range. At the highest Reynolds numbers achieved here, the production and dissipation ranges are separated by an inertial range of about one half decade (Fig. 11). Correlation-coefficient spectra reveal a positive correlation between streamwise and vertical velocity components within the dissipation range, indicating a net transfer of kinetic energy from fluctuations on these scales to the mean flow. The spectral range in which this correlation is found is independent of the mean shear, but the strength of the correlation is controlled by the mean shear and decreases with increasing shear Reynolds number.

Estimates of the turbulent kinetic energy dissipation rate based on a single velocity gradient and the assumption of isotropy are generally accurate for buoyancy Reynolds numbers greater than $O(10^2)$. Estimates based on certain terms, such as $\partial w/\partial z$, are valid down to much lower values of $R_b$. In terms of oceanic observations, these results suggest that vertical profilers equipped to measure vertical velocity may produce particularly good estimates of $\langle \epsilon \rangle$, while horizontal profiling in the direction of the mean current is likely to give serious underestimates of $\langle \epsilon \rangle$ unless $R_b > O(10^2)$. Estimates of $\langle \epsilon \rangle$ based on the assumption of axisymmetry about the streamwise direction are more accurate than some, but not all, of the isotropic estimates tested.

Anisotropy of the scalar gradient indicates a preference for isoscalar surfaces tilted in the direction perpendicular to the mean compressional strain. Estimates of the magnitude of the scalar gradient based on a single component are accurate for Cox numbers greater than $O(10^2)$. Since $R_b = 100$ and Cox = 100 are near the lower end of values considered to be significant in geophysical flows, the effects considered here do not in general present a serious problem for the interpretation of field observations. However, the use of approximate dissipation rates in the analysis of laboratory experiments should be done with careful attention to the potential for error due to small-scale anisotropy.

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