Interaction of Kelvin-Helmholtz and salt sheet instabilities in a sheared, fingering-unstable layer

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Abstract

Motivated by observations of thermohaline staircases in the midlatitude oceans, we have conducted direct numerical simulations (DNS) of the evolution of a diffusively unstable shear layer. This flow supports both KH and salt sheet instabilities, and we wish to see how the two mechanisms work together to drive the transition to turbulence.

The main controlling parameters are the bulk Richardson number $R_i$ and the density ratio $R_\rho$. For oceanic values of these parameters, the linear growth rates of KH and salt sheet instabilities are similar. When the two growth rates are equal, salt sheets dominate, because the spreading of the layer removes the gradients driving shear instability more effectively than it does those driving salt sheets. Nonetheless, over the range of observed parameter regimes, either mode may dominate. At finite amplitude, the system manifests a new secondary instability that we call the “propagating” mode. It is spatially localized and propagates like a bore, though the fluid within it is positively buoyant. Together with the known secondary instabilities of salt sheets and KH billows, this mechanism leads the flow to a fully turbulent state.

1 Introduction

Thermohaline staircases are sequences of well-mixed layers separated by much thinner layers of strong thermal and saline stratification. These structures are commonly observed in the midlatitude oceans (Gregg and Sanford, 1987; St.Laurent and Schmitt, 1999). The density stratification is gravitationally stable but is unstable in the salt-fingering sense, with warm, salty water overlying cool, fresh water (Kunze, 2003). The same mixing processes that create and sustain these structures also mediate the exchange of warm, salty, near-surface waters with cooler, fresher waters of the deep ocean. Thermohaline staircases are generally subjected to some level of current shear due, for example, to the ever-present internal gravity waves. Shear tends to be concentrated in the stratified layers (Gregg and Sanford, 1987). These observations have motivated a sequence of theoretical studies (Smyth and Kimura, 2007; Kimura and Smyth, 2007; Kimura et al., 2010) in which a single interface is modeled as a stratified shear layer of hyperbolic tangent form with thermohaline stratification in the salt fingering regime. The well-mixed layers above and below the interface are represented by the nearly-constant outer limits of the hyperbolic tangent
To simplify the complex interaction of double diffusive and shear-driven mechanisms, we have focused so far on what we call the “weak shear” case. Weak shear renders salt fingers two dimensional (Linden, 1974) and modifies the secondary instabilities via which they become turbulent, but is not strong enough to produce Kelvin-Helmholtz (KH) instability. This regime is defined by the condition that the minimum Richardson number $R_i$ exceed the critical value $1/4$ (Miles, 1961; Howard, 1961). (As used here, $R_i$ is the minimum value of the gradient Richardson number that characterizes the initial mean flow, defined as the ratio of the central values of the squared buoyancy frequency $N^2$ to the squared shear $S^2$; see section 2.4 for detailed definitions.)

Here, we extend our attention into what we will call the “moderate shear” regime $0.18 \leq R_i \leq 0.25$. In these flows, shear is strong enough to produce the basic KH instability, but not strong enough to produce the sub-harmonic pairing instability (Collins and Maslowe, 1988; Klaassen and Peltier, 1989) found at smaller $R_i$. In this moderate shear regime, with density ratio $R_\rho$ (minus the ratio of thermal to saline buoyancy stratification; see section 2.4) near the common value 2, both KH and salt sheet modes are unstable. Our goal here is to explore the various ways the two instabilities can interact as they grow to nonlinear amplitude and become turbulent.

Figure 1 shows some central elements in the interaction between salt sheets and billows in the nonlinear regime. The vertical motions of the salt sheets (arrows at the right) displace and distort the stratified shear flows that the billows grow on, causing adjacent billows to evolve differently. The resulting spanwise velocity gradients exert a frictional damping effect on the billows. Simultaneously, rotational motion in the billow cores decorrelates the vertical buoyancy gradients needed for salt sheet growth. If the salt sheets are not strong enough to shear the billow apart, they will eventually become mixed within it. Separating the billows are ribbon-like features called braids. These are sites of intense strain (arrows at left) which results in sharp, coherent gradients. Salt sheets grow rapidly in these regions and are then advected toward the cores.

Eventually, the flow reaches a turbulent state. In this paper, we use direct numerical simulations (DNS) to explore the complex interactions between salt sheets and KH billows that lead to turbulence. Our methodology is described in section 2. In section 3, we review pertinent aspects of the primary KH and salt sheet instabilities. Section 4 describes the various and complex ways salt sheets and billows can interact in the nonlinear regime. In section 5, we discuss sensitivity of the results to certain parameters whose oceanic values are difficult to reproduce in DNS. Conclusions are given in section 6.
2 Methods

2.1 Mathematical model

The density of salt water is controlled by two scalar concentration fields, temperature and salinity, via an empirically determined equation of state (e.g., Gill, 1982). Buoyancy \( b \) is defined with respect to the characteristic density \( \rho_0 \) and gravitational acceleration \( g \):

\[
b = -g(\rho - \rho_0)/\rho_0.
\]

Neglecting nonlinearities in the equation of state, we assume that the total buoyancy is a simple sum of thermal, \( b_T \), and saline, \( b_S \), contributions:

\[
b = b_T + b_S. \tag{1}
\]

Each component of buoyancy is governed by an advection-diffusion equation. Thermal and saline diffusivities are uniform and are given by \( \kappa_T \) and \( \kappa_S \), respectively:

\[
\frac{Db_T}{Dt} = \kappa_T \nabla^2 b_T; \quad \frac{Db_S}{Dt} = n_S \nabla^2 b_S. \tag{2}
\]

The velocity field \( \vec{u}(x, y, z, t) = \{u, v, w\} \) is nondivergent and is measured in a nonrotating, Cartesian coordinate system \( \{x, y, z\} \). The flow is forced by the gradient of a Boussinesq pressure anomaly, written in the scaled form \( \pi = p/\rho_0 \), and in the vertical by the net buoyancy \( b \). We neglect inertial effects of buoyancy variations in accord with the Boussinesq approximation. Like the buoyancy components, velocity is diffused by a Laplacian operator and a uniform molecular diffusivity (or kinematic viscosity) \( \nu \). The resulting equations are:

\[
\nabla \cdot \vec{u} = 0
\]

\[
\frac{D\vec{u}}{Dt} = -\nabla \pi + \hat{k} + \nu \nabla^2 \vec{u}. \tag{3}
\]

The unit vector \( \hat{k} \) points opposite to gravity.

Boundary conditions are periodic in the horizontal:

\[
f(x + L_x, y, z, t) = f(x, y + L_y, z, t) = f(x, y, z, t). \quad \text{Upper and lower boundaries at} \quad z = \pm L_z/2 \quad \text{are impermeable} \quad (w = 0) \quad \text{and flux free} \quad (\partial u/\partial z = \partial v/\partial z = \partial b_T/\partial z = \partial b_S/\partial z = 0).
\]

2.2 Numerical methods

The numerical code used to solve (1) – (3) is a modified version of that described by Winters et al. (2004). It uses Fourier pseudospectral discretization in all three dimensions, with time stepping via the third-order Adams-Bashforth operator (e.g., Ferziger and Peric, 1996). Viscous and diffusive terms are integrated exactly. The slowly diffusing scalar (salinity) is resolved on a fine grid with spacing equal to one half the spacing used to resolve the other fields (Smyth et al., 2005).

2.3 Initial conditions

Initial mean profiles represent two homogeneous layers separated by a horizontal transition layer of thickness \( 2h \):

\[
B_T(z) = \Delta B_T \tanh \left( \frac{z}{h} \right)
\]

\[
B_S(z) = \Delta B_S \tanh \left( \frac{z}{h} \right)
\]

\[
U(z) = \Delta u \tanh \left( \frac{z}{h} \right), \tag{4}
\]

where \( \Delta B \) and \( \Delta u \) are the half-changes in net background buoyancy and background velocity, respectively, across the transition layer.

This background flow is supplemented by a small perturbation added for computational efficiency. The perturbation has two parts. The first is the sum of the fastest-growing eigenfunctions of the Kelvin-Helmholtz and salt sheet mode from linear theory (Smyth and Kimura, 2007). These are normalized such that the maximum vertical displacement of each is \( 0.02h \). This value was arrived at after significant trial and error. Aside from viscous effects, the results are independent of this amplitude provided it is sufficiently small. Amplitudes much smaller than \( 0.02h \) allow viscosity to alter the mean flow before the instability grows, while values much larger generate artifacts from the nonlinear terms resulting in amplitude dependence. The second perturbation is a random velocity field focussed in a thin layer surrounding the center of the sheet. This is needed to initialize subharmonic secondary instabilities of the salt sheets, but is made as small
as possible so as not to obscure the interactions of the primary KH and salt sheet modes. The amplitude chosen was such that the maximum value of any velocity component is $10^{-5} \Delta u$. This random velocity field adjusts to a nondivergent state after the first time step.

### 2.4 Parameter values

Our equations, initial conditions (disregarding the initial perturbations) and boundary conditions contain ten parameters: \{ν, κ_T, κ_S, h, ∆B_T, ∆B_S, ∆u, L_x, L_y, L_z\}. To reduce this number, we nondimensionalize the problem using $h$ as the length scale and $\sqrt{h/\Delta B}$ as the time scale. (The latter is proportional to the buoyancy period.) The nondimensionalized system has eight parameters, which can be written as \{ν, Pr, τ, Re, ∆u, L_x/h, L_y/h, L_z/h\}. The buoyancy ratio $Re$ is defined as

$$Re = \frac{\Delta u h}{\nu}.$$  

A conceptually useful alternative to the Reynolds number is the ratio of sheet thickness to finger width, which we call the aspect ratio, $\mu$. Using the “tall fingers” approximation (e.g., Smyth and Kimura, 2007), the salt sheet wavelength is $\lambda_s = 2\pi h/(Re^2 Ri Pr)^{1/4}$, so that

$$\mu = \frac{2h}{\lambda_s} = Re^{1/2} Ri^{1/4} Pr^{1/4}/\pi. \quad (9)$$

We further reduce the number of variable parameters by choosing the domain lengths in terms of the known geometry of KH and salt sheet instabilities (e.g. Smyth and Kimura, 2007). For the present experiments, $L_x$ is set to $2\pi h/0.47$, a very close approximation to one wavelength of the KH instability. The domain height is chosen to be large enough to have no significant influence on the results; we have found that $L_z = L_x$ is sufficient. $L_y$ is usually chosen so as to accommodate four wavelengths of the salt sheet instability: $L_y = 4\lambda_s$. (Sensitivity tests with two and eight wavelengths have convinced us that this value is appropriate.)

With these choices, any solution quantity $f$ can now be expressed in nondimensional form as

$$f^* = \frac{f}{h^a \Delta u^b} = f^*(Re, Ri, R_p, \mu, Pr, \tau), \quad (10)$$

where the rational numbers $a$ and $b$ are chosen to give the appropriate dimensions for $f$. The semicolon in (10) isolates $Pr$ and $\tau$. These are chemical properties of salt water that vary little over the scale of a thermohaline staircase and are therefore taken to be constants. Typical oceanic values are $Pr = 7$ and $\tau = 0.01$. In the present simulations, we choose $Pr = 7$ and $\tau = 0.04$. The latter choice is a compromise due to computer hardware limitations. Where appropriate, we attempt to assess the difference that reduction of $\tau$ to the realistic value 0.01 would make. The aspect ratio $\mu$ is also limited by computational resources. Its value is expected to have little influence when $\mu \gg 1$. Here, we choose $\mu = 10.4$. Again, we will attempt to assess the impact of finite $\mu$ on our results.

Our main interest is in the impact of the remaining two parameters, $Re$ and $R_p$. A well-known necessary condition for KH instability is $Re < 0.25$ (Miles, 1961;
Howard, 1961), and the growth rate can be approximated an $\sigma_{KH} \approx 0.2S(1 - 4Ri)$. The condition $Ri < 0.25$ is rarely observed, although KH instability is common (Gregg, 1987; Thorpe, 1987; Smyth et al., 2001). It seems likely that small $Ri$ values are either overestimated due to sampling limitations or are short-lived, i.e., increased by mixing due to the resulting instability before they can be measured. The condition for salt fingering is considerably more relaxed: $1 < R_\rho < \tau^{-1}$. This condition is satisfied over large volumes of the ocean interior (You, 2002). In fingering-favorable oceanic regimes, the density ratio is typically 1.3–3.0.

For the density ratio, we use $R_\rho = 1.6$, 2.0, 3.0 and 25.0. The final value was chosen equal to $1/\tau$ to exclude the primary salt sheet instability. For $Ri$, we use the values 0.18, 0.25 and 0.50. The latter two values exclude KH instability.

### 2.5 Diagnosis via partial kinetic energies

Our main interest is in the way KH and salt sheet modes interact. A difficulty is that, beyond the linear regime, there is no rigorous distinction between these motions. Here we describe a partial decomposition of the kinetic energy that allows us to quantify the amplitudes of the “billow-like” (periodic in $x$, independent of $y$) and “sheet-like” (periodic in $y$, independent of $x$) elements of the flow. Consider the variable $f(x, y)$, which could stand for any component of velocity, buoyancy or pressure. (Dependences on $z$ and $t$ are suppressed for economy.) We decompose $f$ as follows:

$$f - \overline{f}^{xy} = f^{(x)}(x) + f^{(y)}(y) + f_R(x, y).$$  \hspace{1cm} (11)

The overbar indicates an average over the dimensions indicated; for example, $\overline{f}^{xy}$ is the usual horizontal mean. We now define $f^{(x)}(x)$ and $f^{(y)}(y)$ so that $\overline{f^{(x)}(x)} = 0$ and $\overline{f^{(y)}(y)} = 0$. These characteristics typify KH billows and salt sheets early in the flow evolution. At early times $\overline{f}^{xy}$ is the initial mean flow (4), and $f_R$ is the random perturbation, which has the properties $\overline{f_R(x, y)} = 0$. Because of the latter property, we can solve for $f^{(x)}$ and $f^{(y)}$ by averaging:

$$f^{(x)}(x) = \overline{f^{(x)}(x) - \overline{f^{(x)}(x)}} = \overline{f^{(x)}(x) - f(x)} = f^{(x)}(x) = \overline{f^{(x)}(x) - f^{(x)}(x)}.$$ \hspace{1cm} (12)

We define specific kinetic energies associated with billows and salt sheets as

$$K^{(x)}(x) = \frac{\overline{u^{(x)}(x) - \overline{u^{(x)}(x)}}}{2}, \quad K^{(y)}(y) = \frac{\overline{u^{(y)}(y) - \overline{u^{(y)}(y)}}}{2}.$$  \hspace{1cm} (13)

Early in the flow evolution, $K^{(x)} + K^{(y)}$ accounts for the entire kinetic energy. Later, though, the averaging properties of $f_R$ are lost and the decomposition is not complete. We therefore use a combination of flow diagnostics to ensure proper interpretation.

### 3 The “pure” cases: KH billows and salt sheets

In figure 2, we show growth rate curves for the KH (blue) and salt sheet (red) modes along two lines on the $Ri - R_\rho$ plane. These were computed using the methodology described in Smyth and Kimura (2007). In all cases the growth rate was purely real. Line $\overline{AD}$ starts at a point where $Ri = 0.18$, low enough to permit KH instability, but $R_\rho = 25$, marginally too high to allow salt sheets. We then reduce $R_\rho$ through a sequence of values ending at 1.6 (point D). The KH growth rate is essentially independent of $R_\rho$ and therefore does not vary, but the SS growth rate increases monotonically. At the second-to-last value, $R_\rho = 2$ at point C, the growth rates of the two modes are equal, and at point D the salt sheet mode grows faster.

Along line $\overline{DF}$, $R_\rho$ is held constant at 1.6 while $Ri$ is increased. The change in $Ri$ has no effect on the salt sheet growth rate, but causes the KH growth rate to drop to zero near $Ri = 0.25$ and to remain there. We therefore expect that only salt sheets will grow in cases $E$ and $F$.

In the remainder of this section, we will examine first the extreme points $A$ and $F$ at which only one instability can grow, then an intermediate point where KH and salt sheet dynamics are both active. In addition to the primary instabilities, we will review the secondary instabilities that
Table 1: Parameter values for all cases. In every case \( Pr = 7 \), \( Ly = 4\lambda_s \) except for Ct04 and Ct01, where \( Ly = 2\lambda_s \).

<table>
<thead>
<tr>
<th>Label</th>
<th>( Ri )</th>
<th>( R_p )</th>
<th>( \mu )</th>
<th>( \tau )</th>
<th>fine grid size</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.18</td>
<td>25.0</td>
<td>10.4</td>
<td>0.04</td>
<td>( 512 \times 64 \times 1024 )</td>
</tr>
<tr>
<td>B</td>
<td>0.18</td>
<td>3.0</td>
<td>10.4</td>
<td>0.04</td>
<td>( 512 \times 64 \times 1024 )</td>
</tr>
<tr>
<td>C</td>
<td>0.18</td>
<td>2.0</td>
<td>10.4</td>
<td>0.04</td>
<td>( 512 \times 64 \times 1024 )</td>
</tr>
<tr>
<td>D</td>
<td>0.18</td>
<td>1.6</td>
<td>10.4</td>
<td>0.04</td>
<td>( 512 \times 64 \times 1024 )</td>
</tr>
<tr>
<td>E</td>
<td>0.25</td>
<td>1.6</td>
<td>10.4</td>
<td>0.04</td>
<td>( 512 \times 64 \times 1024 )</td>
</tr>
<tr>
<td>F</td>
<td>0.50</td>
<td>1.6</td>
<td>10.4</td>
<td>0.04</td>
<td>( 512 \times 64 \times 1024 )</td>
</tr>
<tr>
<td>Cm8</td>
<td>0.18</td>
<td>2.0</td>
<td>7.8</td>
<td>0.04</td>
<td>( 512 \times 80 \times 1024 )</td>
</tr>
<tr>
<td>Cm16</td>
<td>0.18</td>
<td>2.0</td>
<td>15.6</td>
<td>0.04</td>
<td>( 768 \times 64 \times 1536 )</td>
</tr>
<tr>
<td>Ct04</td>
<td>0.18</td>
<td>2.0</td>
<td>10.4</td>
<td>0.04</td>
<td>( 512 \times 32 \times 1024 )</td>
</tr>
<tr>
<td>Ct01</td>
<td>0.18</td>
<td>2.0</td>
<td>10.4</td>
<td>0.01</td>
<td>( 1024 \times 64 \times 2048 )</td>
</tr>
</tbody>
</table>

Figure 2: Growth rate curves for salt sheet (red) and Kelvin-Helmholtz (blue) modes normalized by the buoyancy frequency and plotted as functions of \( Ri \) and \( R_p \). The Prandtl number, the inverse Lewis number and the aspect ratio are fixed at \( Pr = 7 \), \( \tau = 0.04 \) and \( \mu = 10.4 \), respectively. Vertical dotted lines indicate values used for DNS.

The KH mode

Figure 3 shows the evolution of the partial kinetic energies for case A, in which \( R_p \) is set equal to \( 1/\tau = 25 \), too large to allow salt sheets to grow. Initially, \( K^{(x)} \) grows exponentially at the rate predicted for KH instability. The component \( K^{(y)} \) that would identify salt sheets does not grow. Shortly after the KH billow attains its maximum amplitude, sudden growth of \( K^{(y)} \) heralds the appearance of a secondary instability. This growth continues until about \( N_0 t = 120 \). The saline buoyancy at this stage (figure 4a) shows one wavelength of the KH billow core and braid. The spanwise velocity (figure 4b) has the form of the Klaassen-Peltier mode of secondary instability, which is a standard property of KH billows (Klaassen...
Figure 4: Sample fields from case A. The vertical range shown is $-3/16 \leq z/L_z \leq 3/16$. (a) Saline buoyancy at $N_0t = 120$. Colors range from $-0.6\Delta B_S$ (purple) to $0.6\Delta B_S$ (red); values outside this range are rendered transparent. (b) Spanwise velocity $v$ at $N_0t = 120$, showing Klaassen-Peltier instability. Yellow to red indicates motion to the right with $v \geq 0.03\Delta u$; blue to purple indicates $v \leq -0.03\Delta u$. and Peltier, 1991). This secondary instability is the source of the growth in $K^{(y)}$ seen in figure 3.

3.2 The salt sheet mode

At point $F$ on figure 2, $Ri$ is too high to permit KH instability and only salt sheets grow. Initially, $K^{(y)}$ (figure 5) grows at the rate predicted for salt sheets by linear theory while $K^{(x)}$ remains small. The saline buoyancy field (figure 6) shows four salt sheets as expected. The tips exhibit a weak subharmonic mode with twice the spanwise wavelength of the salt sheets Kimura and Smyth (2010). Beyond $N_0t = 30$, we observe a dramatic increase in $K^{(x)}$. This is not due to a belated KH instability, but rather signals a new instability focused near the tips of the salt sheets (figure 6a). This is likely an extension of the “tip mode” noted by Kimura and Smyth (2007) in cases with much weaker background shear. By $N_0t = 35$ (figure 6b), the new mode exhibits a broadband nature. The shortest wavelength features resemble hairpin vortices. Also evident in the central region is the instability first noted by Holyer (1984) for two-dimensional

Figure 5: Partial kinetic energies for case F ($Ri = 1.6$, $Ri = 0.50$). Short lines at the left indicate exponential growth rates for the salt sheet mode (dotted) and the KH mode (solid) derived from linear theory. Annotations indicate rapid growth of the primary SS mode and the secondary tip mode.

Figure 6: Sample saline buoyancy fields from case F. (a) $N_0t = 30$. (b) $N_0t = 35$. The vertical range shown is $-1/4 \leq z/L_z \leq 1/4$. Colors range from $-0.6\Delta B_S$ (purple) to $0.6\Delta B_S$ (red); values outside this range are rendered transparent.
4 Interaction of KH billows and salt sheets: beyond the linear regime

We begin this section with a close look at case C, for which the primary KH and salt sheet instabilities have equal growth rates. We then expand our view to see how the results change at different $R_{\rho}$ and $R_i$.

4.1 The case of equal growth rates: $R_i = 0.18$, $R_{\rho} = 2$.

Figure 7 shows the kinetic energy components as functions of time for the case $R_i = 0.18$, $R_{\rho} = 2$. In this case, salt sheets and KH billows have equal growth rates (section 3), and we therefore expect that the two modes will interact in interesting ways. In the linear regime (about $0 < N_0 t < 15$), both components are present and grow at the same rate as predicted. As the disturbance reaches nonlinear amplitude, the KH component (solid curve) is damped as the salt sheets (dashed) continue to grow. Figure 8 shows the salinity field just after the billows reach their maximum amplitude.

Beyond $N_0 t = 35$, $K_x$ once again begins to grow, and $K_y$ peaks at $N_0 t = 50$. At this point, the salt sheets are clearly dominant in the transition layer (figure 9a) and are manifesting the characteristic zig-zag instability (right-hand face of the rendered volume). Above and below this layer are wavelike features that propagate with the shear flow, creating an oscillatory instability reminiscent of the Holmboe mode (e.g., Holmboe, 1962; Smyth et al., 2007). Also evident in this region are small scale structures that have the appearance of hairpin vortices.

Upon comparison of figures 3 and 7, it is evident that...
the presence of salt sheets has a significant effect on the evolution of KH billows. As a first approximation, one may imagine the salt sheets “slicing” the billows by advecting them alternately up and down at different cross-stream locations, as is illustrated in figure 1. This generates shear between “sliced” billows growing at different spanwise locations, and hence an increased viscous flux that ultimately damps the billows (figure 7, decrease in $K(x)$ between $N_0t = 25$ and $N_0t = 38$).

If the salt sheets were infinitely tall, this simple picture would describe the effect quite thoroughly. In the present case, however, the salt sheets have tips beyond which their influence decreases. Buoyant fluid within each sheet is compressed against the ambient fluid, generating strong vertical gradients in temperature, salinity and horizontal velocity. The net result of these gradients is to increase shear and reduce net stratification (i.e., reduce the Richardson number), as shown in figure 10. Therefore, as a salt sheets grows, the region within it where $S^2 > 4N^2$, or $Ri < 1/4$, is displaced vertically away from the center of the transition layer and into regions where the background current is nonzero (arrows on figure 10). This promotes the growth of unstable modes with nonzero phase velocity: rightward (leftward) in upgoing (downgoing) salt sheets. The preference for propagating instabilities increases as the salt sheets grow, both because the gradients at the tips become stronger and because the unstable regions are displaced further from the center of the transition layer where the horizontal velocity is greater.

Figure 8 shows both the original KH billow and the right-propagating secondary instabilities at the tips of the upgoing salt sheets. The propagating instabilities interact both with the original billow and with each other, causing the oscillations in $K(x)$ visible in figure 7 between $N_0t = 35$ and $N_0t = 65$. During this time, the propagating mode also grows exponentially at a rate comparable to the salt sheets and the original KH instability. As the salt sheets grow, successive propagating modes build on each other. Because the tips of the propagating regions are localized in $x$ (figure 8), they can excite further instabilities on a wide range of horizontal scales (figure 9).
4.2 Dependence on $R_\rho$ and $Ri$

The effect of salt sheets on billow growth is shown quantitatively by the examples in figure 11. In all cases, $Ri = 0.18$, so the linear growth rate of the KH mode is the same. Accordingly, $K^{(x)}$ (figure 11a) grows initially at the same rate for all four cases, but diverges around $N_0t = 15$. The four cases have widely varying $R_\rho$, and the initial growth rate of salt sheets varies likewise. This is illustrated by $K^{(y)}$, as shown in figure 11b. Case A (dotted) has $R_\rho = 25$, too large to allow salt sheet growth, so $K^{(y)}$ does not grow until $N_0t = 50$, when KP instability of the KH billow appears. At successively lower $R_\rho$, $K^{(y)}$ grows at successively higher rates and peaks at earlier times.

Figure 11: Kinetic energy components for cases A,B,C and D. (a) $K^{(x)}$, initially representing KH billows. (b) $K^{(y)}$, initially representing salt sheets.

In case B, with $R_\rho = 3$, the linear growth rate of the salt sheets is significantly smaller than that of the KH instability. Salt sheets do not reach significant amplitude until well after the KH billow has equilibrated, and the evolution of the latter is therefore hardly changed from the case without salt sheets (cf. dash-dotted and dotted curves on figure 11a). The salinity field at $N_0t = 53$ (figure 12a) shows a well-developed KH billow. Salt sheets are evident at the edge of the billow core, and the propagating secondary instability is also visible, but in general KH dynamics are more dominant than in the $R_\rho = 2$ case. Later in the simulation (figure 12b), salt fingering is active above and below the transition layer, but the KH billow is still very much in evidence.

Figure 12: Saline buoyancy field for case B, with $Ri = 0.18$, $R_\rho = 3$. (a) $N_0t = 53$. (b) $N_0t = 117$. The vertical range shown is $-3/16 \leq z/L_z \leq 3/16$. Colors range from $-0.4\Delta B_S$ (purple) to $0.4\Delta B_S$ (red); values outside this range are rendered transparent.

In case C, $R_\rho = 2$ and salt sheets grow rapidly enough to reverse the growth of the billows around $N_0t = 25$ (figure 11a, dashed curve). After $N_0t = 35$, $K^{(x)}$ begins to grow once again, due not to KH instability but to the propagating instability discussed in the previous subsection (as is evident from the oscillatory growth pattern). As we saw in section 4.1, the growth rates of the KH and salt sheet instabilities are the same in this case, but salt sheet dynamic eventually dominates.

In the final case, $R_\rho = 1.6$ (solid curve on figure 11a), salt sheets dominate throughout. The growth rate of the salt sheets is faster than that of the billows, and the propagating secondary instability appears even before the decay of the KH billows is apparent, as seen from the rapid increase in the slope of the solid curve around $N_0t = 25$. 

10
Next we examine the influence of $Ri$. In figure 13, we compare the kinetic energy evolution for case D (with $R_p = 1.6, Ri = 0.18$) and cases E and F, in which everything is the same but $Ri$ is increased to 0.25 and 0.50, respectively, to suppress KH instability. In case D, the peak in $K(x)$ at $N_0t = 18$ represents KH instability. The KH mode is later overwhelmed by the propagating secondary instability. The presence or absence of KH instability has very little effect on salt sheets or on the resulting turbulence, as can be seen from the fact that $K(y)$ is unchanged late in the simulations and $K(x)$ is the same throughout. See Smyth and Kimura (2010) for a full discussion of the turbulent regime.

### 4.3 Dependence on $\tau$ and $\mu$

Our concern so far has been with the effects of varying $R_p$ and $Ri$, the two measurable parameters used commonly to differentiate mixing driven by shear and salt fingering (St.Laurent and Schmitt, 1999; Inoue et al., 2008). Of the several other parameters whose effects we have neglected, the most important are the diffusivity ratio $\tau$ and the aspect ratio $\mu$, both of which are constrained by computational hardware limitations. Here we describe a brief sequence of experiments designed to provide a glimpse of the results when $\tau$ and $\mu$ take more realistic values.

The effect of $\tau$ is straightforward: reducing its value enhances double diffusive instabilities. As shown in figure 14a, the kinetic energy of the KH mode is initially insensitive to $\tau$. In contrast, salt sheets (figure 14b) grow markedly faster and attain higher amplitudes when $\tau$ is reduced. This leads to a reduction in the maximum amplitude of the KH mode (figure 14a, $N_0t \sim 25$), but a strengthening of the spanwise disturbance that results from the propagating secondary instability (figure 14a, $N_0t > 45$).

The effect of $\mu$ is less obvious. Recall that $\mu$ is the ratio of layer thickness to salt sheet wavelength, and is proportional to $Re^{1/2}Ri^{1/4}$. Figure 15 shows the partial kinetic energies for three cases. The dashed curve represents case C, which we have examined in detail and which has $\mu = 10.4$. For the dash-dotted curve, $\mu$ is reduced to 7.8; for the solid curve, $\mu$ is increased to 15.6. The early evolution of $K(y)$, representing salt sheets, is insensitive to $\mu$, while the growth rate of the primary KH instability increases at higher $\mu$. In other words, changes in $\mu$ shift the flow away from the regime where the linear growth rates of KH and salt sheet instabilities are similar. In the case with reduced $\mu$, KH instability is weak and is damped quickly by the salt sheets, which then grow

Figure 13: Partial kinetic energies for cases D ($R_p = 1.6, Ri = 0.18$), E ($R_p = 1.6, Ri = 0.25$) and F ($R_p = 1.6, Ri = 0.50$). (a) $K(x)$, initially representing KH billows. (b) $K(y)$, initially representing salt sheets.

Figure 14: Partial kinetic energies $K(x)$ (a) and $K(y)$ (b) for two cases with different values of the diffusivity ratio: $\tau=0.04$ (black, case C004) and 0.01 (blue, case C010).

Figure 15: Partial kinetic energies $K(x)$ (a) and $K(y)$ (b) for three cases: dashed (case C), dash-dotted (case C78) and solid (case C156).
to large amplitude and produce a robust propagating instability (figure 15a). In contrast, when $\mu$ is increased, KH instability grows rapidly and maintains its amplitude, while the growth of salt sheets is arrested at a relatively early stage. The propagating secondary instability is not evident in $K^{(x)}$. Instead, while there is clear salt fingering activity in the outer regions of the transition layer, the central region is dominated by a persistent billow (figure 16). Evidently, higher $\mu$ favors the KH instability, whose effect at large amplitude is to mix out the gradients that sustain salt sheets within the transition layer. Salt sheets are then relegated to the relatively weak double diffusive gradients remaining at the outer edges of the layer.

Figure 15: Partial kinetic energies $K^{(x)}$ (a) and $K^{(y)}$ (b) for three cases with different values of the aspect ratio: $\mu = 7.8$ (black, case Cm8), 10.4 (blue, case C) and 15.6 (red, case Cm16). Annotations show the KH, salt sheet (SS) and propagating (P) modes.

Figure 16: Salinity field for $\mu = 15.6$ (case Cm16) at $N_0t = 56$. The vertical range shown is $-3/16 \leq z/L_z \leq 3/16$. Colors range from $-0.6\Delta B_S$ (purple) to $0.6\Delta B_S$ (red); values outside this range are rendered transparent.

5 Summary and conclusions

We have used direct simulations to explore the sequence of secondary instabilities that leads to turbulence in a layer with stratification unstable to salt fingering ($1 < R_\rho < 1/\tau$) and moderate shear ($Ri < 1/4$, but not by much). We have seen that both salt sheet and KH instability mechanisms operate in this regime. Either can dominate, depending mainly on the values of $Ri$ and $R_\rho$. Each mechanism has one or more characteristic secondary instabilities (zig-zag and tip modes for SS, KP mode for KH), and these various modes can combine at finite amplitude to create new flow features.

In one prominent instability that appears to be new, buoyant fluid near the tips of salt sheets develops a highly localized propagating instability due to the strained currents surrounding the KH billows. This can excite streamwise dependence on a wide range of scales, leading to both oscillatory features on the scale of the billow (possibly a Holmboe-like instability mechanism) and small-scale plumes related to the tip mode (Kimura and Smyth, 2007, 2010).

In the range of $Ri$ considered here, salt sheet dynamics is dominant for $R_\rho = 2$ and smaller. In these cases, KH instability appears as a transient perturbation having little long-term influence. When $R_\rho = 3$ and larger, the KH mechanism is dominant, but double diffusive plumes resembling salt fingers appear at the edges of the transition zone.

Lowering the diffusivity ratio $\tau$ to its oceanic value supports the growth of salt sheets. The growth rate of KH billows is unaffected, but at finite amplitude the stronger salt
sheets act more vigorously to destroy the billows. KH billows tend to reach higher amplitude and last longer when the aspect ratio $\mu$ is increased.

In a separate paper (Smyth and Kimura, 2010), we investigate the statistics of mixing in the turbulent flows that result from the transition processes explored here.

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**References**


