Spectral transfers in two-dimensional anisotropic flow

W. D. Smyth
Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada

(Received 11 June 1991; accepted 9 October 1991)

Nonlinear transfers of kinetic energy and enstrophy in the unidimensional Fourier spectrum that is obtained in a two-dimensional shear flow are investigated. Detailed conservation laws reveal the existence of closed pairs of transfers that involve a source mode, a recipient mode, and an advecting mode, which mediates the interaction but is not directly affected by it. The methodology of spectral transfer diagnosis is applied in the study of two simple, linearly unstable, initially parallel shear flows, namely the Bickley jet and the hyperbolic-tangent shear layer. In each case, the transfer spectra are found to be dominated by wave-mean flow interactions and by downscale cascades that are associated with vortex mergers. Cascades are driven by the advective deformation of small eddies by the large-scale vortices (i.e., filamentation), and appear in the transfer function maps as local transfers driven by nonlocal interactions.

I. INTRODUCTION

Geophysical flows are often dominated by motions that are restricted to two dimensions (e.g., McWilliams'). Important insights into the dynamics of such flows may therefore be gained through the analysis of numerically evolved two-dimensional flow models. Efforts in this direction have tended to focus upon isotropic models that are forced at some nonzero wave number (e.g., Benzi et al.'). However, geophysical fluids also tend to be characterized by large-scale velocity shears, and the resulting anisotropic effects can play a dominant role in governing the flow evolution (e.g., Shepherd, Leblond et al., and Smyth and Peltier'). In this paper we will describe a diagnostic technique that provides detailed information regarding spectral transfers of energy and enstrophy in two-dimensional, incompressible anisotropic flow. The technique is based upon the analysis of a one-dimensional Fourier spectrum, which is obtained by applying a Fourier transform over one coordinate and (ultimately) integrating over the other. This technique arises naturally in the study of initially parallel, time-dependent shear flows, since that class of flows is periodic in one direction only.

The behavior of the spectral transfer functions is of central importance in the construction of analytical theories of turbulence, and it has therefore been the subject of numerous investigations (e.g., Herrin et al. and Ohkitani'). The investigations reported here differ from previous work in two important respects. First, we apply the methodology of spectral diagnosis in the study of flows that are not periodic in every coordinate and are therefore modeled using a semi-spectral representation. Second, we seek additional physical insight by avoiding symmetrization of the transfer functions.

In Sec. II, we will describe the mathematical model upon which our investigations are based and derive from it explicit expressions for the one-dimensional spectral transfer functions. We will then develop the detailed conservation laws that are needed in order to characterize spectral transfers in a rational manner. In Sec. III, the methodology will be employed in the diagnosis of energy and enstrophy transfers in two initially parallel shear flows, namely the Bickley jet and the hyperbolic-tangent shear layer. We will see that transfers at large scales are dominated by wave-mean flow interactions, while transfers at small scales are primarily identifiable as "local" transfers (occurring between pairs of small-scale modes having similar characteristic wave numbers), which are effected by the advection of the small-scale eddies by large eddies. In Sec. IV, we will discuss analogous results that are obtained for higher-dimensional spectra. Our results are summarized and discussed further in Sec. V.

II. THEORETICAL CONSIDERATIONS

We consider a two-dimensional flow that is initially parallel, and define the x and z coordinates as denoting the streamwise and cross-stream directions, respectively. We will refer to these coordinates as the "horizontal" and the "vertical," although they might just as well represent the zonal and meridional directions in a large-scale geophysical flow. Our mathematical model is based upon the two-dimensional hydrodynamical equations, which describe planar motions in a homogeneous, incompressible fluid, viz.,

\[ u_x = -uu_x - wu_z - \pi_x + Du, \]  
\[ u_z = -uw_x - ww_z - \pi_z + Du, \]  
\[ 0 = u_x + u_z, \]

in which \( u \) and \( w \) are the usual horizontal and vertical components of the velocity field, \( \pi \) is the pressure normalized by the (constant) density, and \( D \) is a diffusion operator whose form will be left unspecified for the present. Subscripts denote partial derivatives. On the upper and lower boundaries, we impose conditions of vanishing tangential stress and impermeability:

\[ u_z = w = 0, \quad \text{if} \quad z = 0, H, \]  

\( H \) being the domain height. Since the equations and initial conditions are autonomous with respect to \( x \), we may assume periodicity in that direction:

\[ f(x + L, z, t) = f(x, z, t), \]

in which \( L \) is the domain length and \( f \) may represent \( u, w, \) or...
In the present study, $L$ will be chosen to be an integer multiple of the wavelength of the most unstable eigenmode of the initial shear flow. In consequence of (3), it is natural to expand the solution fields in Fourier series form, viz.,

$$f(x,z,t) = \sum_{\nu=-N}^{N} f_\nu(z,t) e^{i\nu x},$$  

(4)

in which $\alpha = \frac{2\pi}{L}$. Note that $f_{-\nu} = f^*_\nu$ is required in order that the solution fields be real valued. Substituting (4) into (1) and applying the usual Galerkin formalism, we obtain

$$u_{\nu,t} = -\sum_{K} (i\omega \lambda u_{\nu} u_{K} + w_{\nu} w_{K}) \delta_{\nu + \lambda + \alpha,0},$$  

(5a)

$$w_{\nu,t} = -\sum_{K} (i\omega \lambda u_{\nu} w_{K} + w_{\nu} w_{,K}) \delta_{\nu + \lambda + \alpha,0},$$  

(5b)

$$0 = i\omega \nu u_{\nu} + w_{\nu,t},$$  

(5c)

where $\delta$ is the Kronecker delta. For the left-hand sides of (5a) and (5b) we have employed the complex conjugate of the expansion (4), which is permissible since $f(x,z,t)$ is real. In the present discussion we shall be concerned only with the nonlinear advection terms in (1). In (5), and in the remainder of this section, the diffusion and pressure terms that appear in (1) are therefore omitted. Note that the wave-number indices $\nu$ and $\lambda$ have been arranged so that each of the terms on the right-hand sides of (5a) and (5b) may be interpreted as representing the advection of a component field having index $\lambda$ by a component field having index $\nu$. In what follows, we shall refer to the Fourier modes identified by the indices $\nu$ and $\lambda$ as the “advecting mode” and the “advected mode,” respectively.

We define the horizontally averaged, vertically integrated kinetic energy $E_\nu$ associated with mode $\nu$ as

$$E_\nu = c_\nu \int_{0}^{H} dz \frac{1}{2} (u_{\nu}^* u_{\nu} + w_{\nu}^* w_{\nu}),$$  

(6a)

$$c_\nu = \begin{cases} 1, & \text{if } \nu = 0, \\ 1, & \text{if } \nu \neq 0. \end{cases}$$  

(6b)

It is readily shown that $E_\nu$ evolves in accordance with

$$\dot{E_\nu} = \sum_{K} \dot{E}(\nu,K,\lambda),$$  

(7a)

in which the spectral transfer functions $\dot{E}$ are given by

$$\dot{E}(\nu,K,\lambda) = -\text{Real} \int_{0}^{H} dz \ c_\nu (i\omega \lambda u_{\nu} u_{K} + w_{\nu} w_{K})$$

$$+ u_{\nu} w_{,K} + i\alpha \lambda u_{\nu} u_{,K}$$

$$+ w_{\nu} w_{,K}) \delta_{\nu + \lambda + \alpha,0}. $$  

(7b)

Note once again that we are concerned only with the terms that describe advection.

In similar fashion, we define the enstrophy $Z_\nu$ associated with mode $\nu$ as

$$Z_\nu = c_\nu \int_{0}^{H} dz \frac{1}{2} \omega_{\nu}^* \omega_{\nu},$$  

(8)

in which $\omega = \psi_{xx} + \psi_{yy}$ is the vorticity and the streamfunction $\psi$ is defined by $u = -\psi_{z}$, $w = \psi_{x}$. The evolution equation for $Z_\nu$ is

$$Z_\nu = \sum_{\nu} \dot{Z}(\nu,K,\lambda),$$  

(9a)

in which

$$\dot{Z}(\nu,K,\lambda) = -\text{Real} \int_{0}^{H} dz \ c_\nu (i\omega \lambda u_{\nu} u_{K} + w_{\nu} w_{K})$$

$$- \kappa \omega \nu u_{,K} + \delta_{\nu + \lambda + \alpha,0}. $$  

(9b)

The expressions given in (7b) and (9b) will be interpreted as transfers into mode $\nu$ due to the advection of mode $\lambda$ by mode $\kappa$.

Before we proceed to compute $\dot{E}$ and $\dot{Z}$ for particular flows, we will examine the mathematical properties of (7b) and (9b) in detail. Specifically, we must establish the detailed conservation laws, which will allow us to trace the flow of energy and enstrophy from one mode to another in a rational manner. We will begin by discussing the simple case in which one of the interacting modes is mode $0$, i.e., the mean flow. This particular class of interactions is important in the applications to be discussed in Sec. III, since the flows that we will examine are strongly influenced by the dynamical instability of the mean flow. In this discussion, we will focus on energy transfers in particular, and we will assume that the energy transfer into the disturbance is positive. If that quantity is negative, the interpretation is reversed in the obvious fashion. Suppose first that energy is transferred into a disturbance that is represented by mode $\nu = n \neq 0$ via the advective action of the mean flow on the disturbance. Setting $\kappa = 0$ and $-\lambda = \nu = n$ in (7b), we find that $\dot{E}(n,0,-n) = 0$. This process represents the Orr effect, which does not alter the horizontal spectrum (Shepherd) and is therefore transparent to our analysis. If, on the other hand, we set $\lambda = 0$ and $-\kappa = \nu = n$, we obtain the familiar expression

$$\dot{E}(n,-n,0) = -\text{Real} \int_{0}^{H} dz \ u_{n} w_{n0} w_{n0},$$  

(10)

which describes energy transfer into the disturbance via the Reynolds stress. Wave–mean flow interactions thus appear, in the context of the present formalism, as a consequence of the advection of the mean flow by the disturbance.

In order to fully characterize any spectral transfer, we must identify the “source” of the transferred quantity as well as its “destination.” For the simple case of a wave–mean flow interaction, only two modes participate and the mean flow must therefore be the source of the transferred energy. To confirm this, we set $\nu = 0$ and $-\kappa = \nu = n$ in (7b) or

$$\dot{E}(0,-n,n) = -\frac{1}{2} \dot{E}(n,-n,0) = \dot{E}(0,n,-n)$$

or

$$\dot{E}(n,-n,0) + \dot{E}(0,-n,n) + \dot{E}(0,n,-n) = 0.$$  

(11a)

This detailed conservation law states that wave–mean flow interactions are energetically closed. Examining wave–mean flow enstrophy transfers in the same fashion, we find

$$\dot{Z}(n,-n,0) + \dot{Z}(0,-n,n) + \dot{Z}(0,n,-n) = 0.$$  

(11b)

Transfers in which the mean flow does not participate involve not two but three modes, and the forms of the detailed conservation laws that pertain to them are correspondingly less obvious. We will introduce this discussion...
with a brief summary of the detailed conservation laws that are obtained in higher-dimensional spectra. If we imagine a flow that consists instantaneously of only the three modes having wave-number indices \( a, b, \) and \( c \), we may write the evolution equations for the kinetic energy contained in those modes as

\[
\begin{align*}
E_{a,t} &= \dot{E}(a,b,c) + \dot{E}(a,c,b) \quad (12), \\
E_{b,t} &= \dot{E}(b,c,a) + \dot{E}(b,a,c) \quad (1), \\
E_{c,t} &= \dot{E}(c,a,b) + \dot{E}(c,b,a) \quad (2).
\end{align*}
\]

(The significance of the superscripts will be discussed presently.) In the usual formulation of the transfer functions, the right-hand sides of (12) are replaced by single functions, viz.,

\[ T(v,K,\lambda) = \dot{E}(v,K,\lambda) + \dot{E}(v,\lambda,K), \]

so that \( T(v,K,\lambda) \) represents both of the contributions to mode \( v \) that result from interactions with modes \( K \) and \( \lambda \) (e.g., Pedlosky,\(^8\) Herring et al.,\(^6\) and Ohkitani\(^7\)). Energy conservation (neglecting viscosity) therefore requires that

\[ T(a,b,c) + T(b,c,a) + T(c,a,b) = 0 \quad (14). \]

It may be shown (e.g., Herring et al.\(^8\)) that (13) holds for the more general case in which the flow involves other modes in addition to \( a, b, \) and \( c \). If we wish to retain the distinction between the physical processes that are represented by \( \dot{E}(v,K,\lambda) \) and \( \dot{E}(v,\lambda,K) \), then the detailed conservation law that corresponds to (14) must be written in a form that includes six terms. Fortunately, the conservation relations can be obtained in a more specific form, as close inspection of (7b) reveals that

\[ \dot{E}(n,l,k) + \dot{E}(k,l,n) = 0 \quad (15). \]

This is easily demonstrated by performing integrations by parts on the terms in (7b) which involve vertical derivatives and noting that \( \kappa + \lambda = -\nu \), thereby obtaining

\[
\dot{E}(v,K,\lambda) = \text{Real} \int_0^H dz (iavv_1wu_1 + u_{v_1}u_{w_1}w_1 \\
- ivw_1w_{v_1} + w_{v_1}w_{w_1}w_{v_1} \delta_{\kappa + \lambda + \nu_0} \\
= - \dot{E}(\lambda,K,v). \quad (16)
\]

In this calculation, we have assumed that \( \nu \lambda \neq 0 \) and therefore \( c_v = c_{-1} = 1 \), since transfers involving advection of the mean flow have been dealt with previously (and transfers in which \( \kappa = 0 \) vanish). As a result of (13), we see that the terms that appear in (14) cancel in pairs. The superscripts (1), (2), and (3) that appear in (12) have been inserted in order to identify the pairs of terms that cancel.

The detailed conservation relation (15) defines energetically closed pairs of interactions in which a transfer of energy between mode \( n \) and mode \( l \) is effected by the advective action of mode \( k \) on the former two modes. In other words, the transfer into mode \( n \) due to advection of mode \( l \) by mode \( k \) is balanced by a transfer out of mode \( l \) due to the advection of mode \( n \) by mode \( k \). [Again, this interpretation is reversed if \( E(n,k,l) < 0 \).]

The foregoing discussion is readily extended to include enstrophy transfers. In this case we obtain a detailed conservation law for the enstrophy which is analogous to (15), viz.,

\[ \dot{Z}(n,l,k) + \dot{Z}(k,l,n) = 0. \quad (17) \]

Once again, we have assumed that \( \nu \lambda \neq 0 \).

After some experimentation, we have found that the transfer functions are represented most clearly in a coordinate system defined by the ordinate \( \mu = -\kappa + \lambda \) and the abscissa \( \nu = -\kappa - \lambda \). Due to the symmetry of the transfer functions with respect to the transformation \( (\nu,K,\lambda) \rightarrow (-\nu, -\kappa, -\lambda) \), we examine only the region \( \nu > 0 \). The \( \kappa \) and \( \lambda \) axes are then oriented diagonally, as is illustrated in Fig. 1. In the region \( \nu > 0 \), at least one of \( \kappa \) and \( \lambda \) must be negative, and we therefore find it convenient to identify the diagonal axes as \( -\kappa \) and \( -\lambda \). We will now conduct a brief "tour" of the \( \mu-\nu \) plane in order that the reader may become familiar with the manner in which the transfer functions are interpreted.

The \( \mu-\nu \) plane shown in Fig. 1 has been divided into three regions denoted by Roman numerals. The detailed conservation laws (15) and (17) each describe two classes of interactions, which we shall identify as types 1 and 2. In type 1 interactions, a transfer located in region II is balanced by a transfer located in region I. In type 2 interactions, both transfers are located in region III. Interactions involving the mean flow appear on the \( \mu \) and \( -\kappa \) axes. Transfers due to advection of eddies by the mean flow are found on the \( -\lambda \) axis, and since these transfers are transparent to this analysis (see the earlier discussion), the transfer functions vanish on that axis. The self-interaction transfer, in which a mode transfers energy/enstrophy into its first harmonic, is located on the \( \nu \) axis, while the corresponding loss of energy/enstrophy that the former mode experiences due to interaction with the latter is located in region I. This transfer pair, operating in the reverse direction, may be identified with the growth of a subharmonic via resonant interaction with the corresponding fundamental mode (Kelly\(^9\) and Collins and Maslowe\(^10\)), or with the upscale energy cascade, which is characteristic of two-dimensional turbulence (Kraichnan\(^11\)).

The phenomenology of one-dimensional spectral transfers is illustrated further in Fig. 2, in which we show an ordered sequence of transfer pairs, beginning with a type 1 interaction and ending with a type 2 interaction. In the type 1 interactions shown in Figs. 2(a) and 2(b), transfers located in region II are balanced by counterparts in region I. In Fig. 2(a), for example, the solid dot shows mode \( -\kappa = 1 \) advecting mode \( -\lambda = 3 \), an interaction that reinforces mode \( \nu = 4 \). Correspondingly, the circle shows mode 1 (represented now by \( \kappa - 1 \)) advecting mode 4 and thereby weak-
FIG. 2. Representative pairs of energy/enstrophy transfers that are connected by (15) and (17).

III. APPLICATION TO SIMPLE UNSTABLE SHEAR FLOWS

In this section we will employ the methodology of spectral transfer diagnosis that was developed in Sec. II in order to discover the characteristics of the transfers that drive the nonlinear evolution of two simple parallel shear flows. The flow simulations that we will discuss have been conducted using a numerical model, which is described in detail in Smyth and Peltier. Briefly, Eqs. (1) that describe homogeneous, incompressible, two-dimensional flow are cast into vorticity-streamfunction form, and a Laplacian diffusion operator is specified, viz.,

$$\omega_t = -J(\omega, \psi) + \text{Re}^{-1} \Delta \omega, \quad (18a)$$

$$\omega = \Delta \psi, \quad (18b)$$

in which $\omega$ and $\psi$ are the vorticity and streamfunction fields that we defined in Sec. II, $J$ is the Jacobian operator, and Re is the Reynolds number. Exploiting the horizontal periodicity condition (3), (18) is converted into a set of evolution equations for the individual horizontal Fourier modes, just as was done in Sec. I. The nonlinear terms are evaluated in physical space. The $z$ dependence is discretized using second-order centered differences. Time stepping is performed using the leapfrog method, with an occasional Euler backstep to suppress computational instability. The exception to this is the diffusion operator which is stabilized by replacing the leapfrog method with a first-order explicit scheme.

In Fig. 3, we display vorticity contours for an evolving Bickley jet, whose initial condition is given by $u = \text{sech}^2(z - H/2)$. The initial state also includes a small-amplitude disturbance which is proportional to the most-unstable linear eigenmode, as well as a random noise field that exhibits a $k^{-3}$ spectrum in the horizontal. The most unstable eigenmode has horizontal wave number close to unity (e.g., Drazin and Reid). The domain length $L$ is chosen so as to accommodate four wavelengths of this eigenmode, and the latter thus corresponds to the wave-number index $\nu = 4$. The domain height $H$ is set equal to $L$ in order
that the flow evolution will not be significantly influenced by the presence of the upper and lower boundaries. The Reynolds number is set at \( \text{Re} = 500 \), and the spatial discretization is accomplished using 370 grid points in the vertical and 96 Fourier modes in the horizontal.

Upon inspection of the early stages of the evolution displayed in Fig. 3, we see that the unstable disturbance is characterized by periodically spaced vortices of alternating sign and may be identified with the sinuous mode of instability (Drazin and Reid\(^{13} \)). For consistency with results to be presented later in this section, we represent negative (clockwise) vorticity with solid contours and positive (counterclockwise) vorticity with dashed contours. Between \( t = 80 \) and \( t = 100 \), the four clockwise vortices that are present initially merge to form two larger vortices. The filamentation process that accompanies vortex mergers (e.g., Melander \textit{et al.}\(^{14} \)) is visible at \( t = 100 \). The counterclockwise vortices do not merge but rather are extruded into filaments by straining deformations that are associated primarily with the clockwise vortices. This asymmetry in the evolution of the two vortex types is most probably due to a slight asymmetry in the initial noise field, which promotes early merging of the clockwise vortices. The large clockwise vortices that result are relatively resistant to deformation.

Energy and enstrophy transfers are shown in Figs. 4 and 5, respectively. Positive (negative) transfers are indicated by solid (dashed) contours. For plotting purposes, the transfer functions have been scaled according to

\[
\hat{E} \rightarrow \begin{cases} \hat{E}^{1/2}, & \text{if } \hat{E} > 0, \\ \left( \frac{\hat{E}}{\text{Re}} \right)^{1/2}, & \text{if } \hat{E} < 0, \end{cases}
\]

and similarly for \( \hat{Z} \). This has the effect of amplifying the weak transfers among small eddies, which would otherwise not be visible in the diagrams. In all of these representations, the asymmetries predicted by (15) and (17) are clearly evident. At \( t = 20 \) the fundamental disturbance is approaching nonlinear amplitude, and transfers of energy and enstrophy into mode 4 from the mean flow are observed on the \(-\kappa\) axis, as expected. The positive peak in the center of the diagram corresponding to \( t = 20 \) indicates a transfer of energy into mode 8 due to the self-interaction of mode 4. The negative peak located at \( \mu = -12, \nu = 4 \) reveals a corresponding loss of energy experienced by mode 4 due to its interaction with mode 8. Similar transfer pairs are visible at higher values of \( \nu \), and reveal the nonlinear growth of the higher harmonics of mode 4. The interactions described above are all of type 1, and are driven by the advective action of mode 4 upon itself and upon its harmonics. Type 2 interactions are also visible, but are relatively weak. To the right of each contour map shown in Figs. 4 and 5 is a diagram of the corresponding summed transfer function \( T(\nu) \), which is just the sum of the values of the transfer function over all values of \( \mu \). In order that the absolute strengths of the interactions may be accurately assessed, the summed transfer function is not scaled using (19). Upon examination of these diagrams, we see that the transfers of energy originating in the mean flow are, in general, an order of magnitude stronger than any of the wave–wave interactions that are present in this flow.

At \( t = 60 \), we find that the transfer spectrum is dominated by the wave–mean flow interaction transferring energy/enstrophy into mode 2. Also visible near the \(-\kappa\) axis are weak type 1 upscale transfers into mode 4 from modes 5 and 6 that are accomplished by the advection of the latter two modes by modes 1 and 2 (respectively). At \( t = 100 \), we observe that mode 2 is losing energy and enstrophy to the mean flow while mode 1 is gaining energy/enstrophy from that source. In addition to this, the enstrophy transfer spectrum reveals transfers of enstrophy to small scales due to the advection of high-wave-number modes by mode 2. This process is also visible at \( t = 80 \) (not shown) and clearly corresponds to the filamentation that is associated with the consolidation of the mode 2 clockwise vortices (cf. Fig. 3).
In summary, this flow is dominated by two primary transfer types: at large scales the dynamics is dominated by nonlocal wave–mean flow interactions, while at small scales we observe local transfers of enstrophy that are primarily type 1 interactions driven by advection of small eddies by the large-scale flow. The latter process is associated with filamentation that occurs during vortex mergers.

For our second application, we investigate spectral transfers in a shear layer whose initial state is given by $u = \tanh(z - H/2)$. This profile represents a substantially larger reservoir of both kinetic energy and enstrophy than does the initial profile of the Bickley jet. In view of this, we expect that the energy and enstrophy transfers that occur in the shear layer will be stronger than those that are obtained in the jet case. The most-unstable eigenmode of the shear layer is characterized by the horizontal wave number $0.45$. As was done in the previous case, we choose the domain size so as to accommodate four wavelengths of the unstable
mode. The Reynolds number is again set to the value 500, and the flow is resolved using 700 grid points in the vertical and 96 Fourier modes in the horizontal.

The evolution of the vorticity field is illustrated in Fig. 6. As was done in Fig. 3, we employ solid contours to denote negative (clockwise) vorticity. The initial agglomeration of the vorticity contained in the shear layer into discrete vortices is a manifestation of Kelvin–Helmholtz instability. As the flow evolves, we observe two distinct merging events, the first occurring near $t = 60$ and the second near $t = 100$. As the evolution progresses beyond the phase shown in Fig. 6, the vorticity field relaxes to an approximately axisymmetric distribution, further mergings being suppressed by the boundary conditions.

In Fig. 7 we display the enstrophy transfer spectra for the hyperbolic-tangent shear layer. Rather than display the energy transfer spectra for this case, which reveal nothing that is not seen in the enstrophy spectra, we show the latter at six separate times. At $t = 20$, we observe a strong transfer from the mean flow to the Kelvin–Helmholtz wave train (mode 4), accompanied by weaker transfers into the higher harmonics (modes 8, 12, etc.), just as was seen in the early stage of the evolution of the Bickley jet. At $t = 40$, we find that mode 4 is losing enstrophy to the mean flow, while mode 2 is gaining enstrophy from the same source. Also visible at $\mu = -6$, $\nu = 2$ is a positive peak, which indicates an upscale transfer of enstrophy from mode 4 into mode 2. Quantitative analysis reveals that this upscale transfer provides $\sim 20\%$–$30\%$ of the enstrophy that accrues to mode 2 during the merging phase (also see Smyth and Peltier\textsuperscript{5}). This result is in agreement with the prediction of Kelly\textsuperscript{9} who showed that a resonant interaction between the primary Kelvin–Helmholtz wave and its subharmonic may amplify the latter so that its growth rate is increased by $\sim 20\%$ over the value predicted on the basis of linear theory. The corresponding loss of enstrophy accruing to mode 4 is revealed by the negative peak at $\mu = 0$, $\nu = 4$. At $t = 60$, continued growth of mode 2 is observed along with a self-interaction of mode 2, which transfers enstrophy into mode 4. Simultaneously, mode 4 is losing enstrophy to the mean flow via wave–mean flow interaction ($\mu = 4$, $\nu = 4$). The downscale cascade that is associated with vortex pairing is clearly visible at this point in the simulation. For $-\lambda$ less than about 12, these type 1 transfers may be identified with interactions between modes 2 and 4 (i.e., $-\kappa = 2, 4$) and the smaller eddies, whereas transfers involving the smallest eddies are apparently mediated by modes 1 and 3.

At $t = 80$, the largest peak in the enstrophy transfer spectrum indicates growth of mode 1 due to wave–mean flow interaction. The second large peak corresponds to advection of mode 1 by mode 2, which causes enstrophy to be transferred into mode 3. As was seen in the growth phase of mode 2, the growth of mode 1 is reinforced by a weak resonant interaction with its first harmonic. In addition, a significant contribution to the enstrophy contained in mode 1 comes from a type 2 interaction between modes 2 and 3 ($\lambda = 2$, $-\kappa = 3$). The enstrophy transfer spectrum at $t = 100$ is strongly dominated by transfer into mode 1 from the mean flow. The enstrophy cascade that we observed during the first merging event ($t = 60$, $t = 80$) is weakened at $t = 100$, but appears again at $t = 120$ (not shown) and $t = 140$. This second cascade phase clearly corresponds to the second vortex merging event during which mode 1 becomes the dominant feature of the flow. The strongest downscale transfers that we observe during this phase of the flow evolution are driven by the advection of small eddies by modes 2 and 3.

Besides being characterized by strong spectral transfers due to the large amounts of energy and enstrophy that are contained in the initial flow, the shear layer differs qualitatively from the Bickley jet in that it involves only clockwise vorticity. Despite these differences, the general characteris-

---

**FIG. 6.** Contours of the vorticity field for an evolving hyperbolic-tangent shear layer.
tics of the spectral transfers that are observed in the two flows are very similar. In each case, we see a sequence of successively larger-scale vortices growing primarily as a consequence of the broadband instability of the parallel component of the flow (cf. Smyth and Peltier). As each large vortex appears, it absorbs two smaller vortices into its own structure, and we observe simultaneously a cascade of enstrophy to small scales. These cascades appear as sequences of local interactions in which small eddies transfer enstrophy to slightly smaller eddies due to the advective action of the large vortices [cf. Fig. 2(a)]. Nonlocal interactions, in which a transfer of energy/enstrophy from a large eddy into a small eddy is mediated by the advection of the large eddy by another small eddy [cf. Figs. 2(b) and 2(d)], are rarely observed. This is surprising, as one would expect that the deformation of small eddies by large would be accompanied by transfers affecting the latter, and the strong wave–mean flow interactions [Fig. 2(c)] that drive the flows away from the parallel state operate in essentially this manner. In light of the observed similarity between the spectral transfer prop-

FIG. 7. Enstrophy transfer spectrum for the flow shown in Fig. 6. Solid (dotted) contours represent positive (negative) transfer into mode v due to the advection of mode A by mode κ.
erties of the two flows examined here, we speculate that the
properties discussed above may be characteristic of a broad
class of initially parallel shear flows.

IV. CONNECTION WITH RESULTS PERTAINING TO HIGHER-DIMENSIONAL SPECTRA

When the flow is periodic in all directions, a complete
Fourier transformation of the momentum equations is possible, and detailed conservation relations analogous to (15)
and (17) are easily derived. Let the velocity component \( u_i \)
be expanded as

\[
u_i(x) = \sum_k u_i(k) e^{i k x},
\]

in which \( \epsilon = \sqrt{-1} \). The rate of change of the kinetic energy
associated with mode \( k \) due to the nonlinear advection terms
may then be written as

\[
\frac{dE_k}{dt} = \sum_{pq} \tilde{E}(k,p,q); \quad \tilde{E}(k,p,q) = -u_j u_i(k) u_j(p) u_i(q) \delta_{k+p+q,0}.
\]

Interchanging \( k \) and \( q \) and employing the incompressibility
condition \( \epsilon u_i(k) = 0 \), we obtain

\[
\tilde{E}(k,p,q) + \tilde{E}(q,p,k) = 0.
\]

The analogous result for enstrophy transfer is readily de-
In the case of the one-dimensional spectrum (cf. Sec. II), we may identify this transfer as a consequence of the
advection of modes \( k \) and \( q \) by mode \( p \). We emphasize that
the physics that is revealed by (22) is obscured when
\( \tilde{E}(k,p,q) \) is replaced by the symmetrized form
\( T(k,p,q) = \frac{1}{2} [ \tilde{E}(k,p,q) + \tilde{E}(k,q,p) ] \), as is the usual
practice.

V. DISCUSSION

We have described a technique for the detailed study of
spectral transfers of energy and enstrophy in a two-dimen-
sional, anisotropic flow. The absence of periodicity in the
vertical direction precludes the application of a fully two-
dimensional spectral analysis and therefore the study of any
process, which affects only the vertical structure of the flow.
However, this deficiency is important only in the interpreta-
tion of wave-mean flow interactions. Processes that involve
only the nonparallel Fourier components, since they invari-
ably affect the horizontal structure of the flow, are readily
analyzed in the context of the one-dimensional spectrum.
The relevant detailed conservation laws reveal that spectral
interactions involve wave-number triads, each of which is
composed of a source mode, a recipient mode, and an advec-
ting mode, which mediates the transfer but is not directly
affected by it. This transfer structure allows us to define
"local" and "nonlocal" interactions in the context of the
unidimensional spectral analysis and to distinguish between
them in analyzing the results of numerical simulations.

We have employed this diagnostic in the analysis of two
unstable parallel shear flows whose large-scale structures
differ significantly, and have found that the energy and en-
strophy transfer spectra exhibit a considerable degree of si-
ilarity. In each case, we have seen that the strongest trans-
fers usually occur between the mean flow and single nonzero
Fourier component. This process is readily understood in
terms of the broadband linear instability of the parallel com-
ponent of the flow (e.g., Smyth and Peltier). (Inspection of
the Fourier-transformed equations of motion reveals that, in
a flow that consists only of mode 0 plus some nonzero Four-
der component, the disturbance evolves in accordance with
the linearized equations, although the nonlinear back-reac-
tion of the disturbance upon the mean flow is also present in
that model.) In addition to wave-mean flow interactions, we
have observed manifestly nonlinear processes, including the
growth of the harmonics of the original unstable eigen-
mode, accelerated growth of subharmonics due to resonance
with the fundamental (Kelly) and downslope cascades as-
associated with vortex merging (e.g., Melander et al.).

The upscale energy transfers that are observed in a large class of
two-dimensional flows are in the present case overwhelmed
by wave-mean flow interactions. In each of the two flows
that we have examined, transfers to small scales are due al-
most entirely to local transfers that are driven by the advect-
ive action of large vortices upon small-scale features. This
result is consistent with the cascade mechanism, proposed
by Kraichnan (also see Herring), in which the large-
scale strain field transfers energy to small scales by deforming
small eddies.

Domaradski and Rogallo have shown that downslope cascades in two- and three-dimensional iso-
tropic flows are driven primarily by interactions in which
one member of the wave-vector triad is much shorter than
the other two, a result that is clearly similar to the results
that have been discussed in Sec. III of the present paper.
Domaradski and Rogallo have made a particularly valu-
able contribution to the terminology of triad interactions by
emphasizing the distinction between local (or nonlocal)
transfers and local (or nonlocal) interactions. The picture
that emerges from these studies is one in which cascades are
controlled by local transfers that are driven by nonlocal in-
teractions. The negative implications of these results for the
validity of Kolmogorov-type scaling theories are obvious,
since those theories depend upon the independence of small
scales from large scales (i.e., upon the dominance of local
interactions in the inertial range). However, the results that
we have described in Sec. IV indicate that small-scale dy-
namics may be quite insensitive to the particular structure of
the large-scale flow, even though the two are strongly
connected.

ACKNOWLEDGMENTS

The author has benefitted from enlightening discussions with Tom Warn during the preparation of this paper. Com-
putational resources were kindly provided by the Ontario Center for Large-Scale Computing at the University of
Toronto.