Efficiency of shear-driven mixing forced by internal waves

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ABSTRACT

The dependence of mixing efficiency on time-varying forcing is studied by direct numerical simulation (DNS) of Kelvin-Helmholtz (KH) instability. Time-dependent forcing fields are designed to mimic a breaking internal wave by solving the equations of motion in a tilted coordinate frame and allowing the tilt angle to vary in time. Mixing efficiency $\Gamma_c$ is defined as the ratio of potential energy gain to dissipation, both averaged over one wave cycle and examined via parameters representing forcing waves; minimum Richardson number $Ri_{\text{min}}$ and normalized frequency of the forcing $\omega/N$. The effect of Reynolds number $Re_0$ and the initial random disturbance amplitude $b$ are also examined. In our experiments, $\Gamma_c$ varies between 0.21 and 0.36 and is controlled by the timing of two events; the emergence of KH billows and arrival of the deceleration of the mean shear by the wave forcing. $\Gamma_c$ is higher than canonical value of 0.2 when the deceleration phase of the wave suppresses less efficient turbulence after breakdown of KH billows. However, when $Ri_{\text{min}}$ and $\omega/N$ are small, KH billows start to develop before $Ri_{\text{min}}$ is achieved. Therefore, the forcing accelerates mean shear and thereby sustains turbulence after breakdown of KH billows. The canonical value is then reproduced in the DNS. Although larger values of $Re_0$ and $b$ intensify the development of KH billows and modify $\Gamma_c$, this effect is less significant when forcing fields act to sustain turbulence. The time averaged Thorpe scale and Ozmidov scale are also used to see how mixing is modified by forcing fields and compared with past microstructure measurements. We find that DNS also corresponds to past observations if the forcing accelerates mean shear to sustain turbulence.
1. Introduction

Internal waves are the main cause of turbulence in the ocean thermocline and the deep ocean (Munk 1981; Gregg 1987; Garrett and St. Laurent 2002). Wave-wave interaction theory and microstructure measurements have shown that the turbulent kinetic energy dissipation rate can be estimated from the 10m vertical shear (e.g. Gregg 1989; Polzin et al. 1995; Sun and Kunze 1999). However, the mixing rate depends further on mixing efficiency, $\Gamma$, or buoyancy flux as a fraction of the energy dissipation rate (e.g. Osborn 1980). Osborn (1980) determined $\Gamma$ to be 0.2 assuming that the steady-state balance is achieved in the ocean mixing caused by shear instability and that flux Richardson number ($R_f$) is restricted to critical value to maintain steady-state turbulence. Here, $R_f$ is defined as the ratio of potential energy gain to kinetic energy gain and related to $\Gamma$ by $\Gamma = R_f/(1 - R_f)$ in steady state turbulence. The value 0.2 is often used in the ocean modeling as well as in interpretation of microstructure measurements (Munk and Wunsch 1998; Wunsch and Ferrari 2004; Simmons et al. 2004; Polzin et al. 1997; Arneborg 2002), but it is likely to depend on various factors, such as the duration and intensity of high shear events which lead to instability and turbulence (Garrett 2001).

Because of its importance, mixing efficiency for shear-driven turbulence has been studied through in-situ ocean observations, laboratory experiments, theories, and numerical simulations. In the ocean observations, Oakey (1982; 1985) defined $\Gamma$ in stratified steady turbulence which can be estimated from microscale temperature-gradient and velocity shear measurements. Ivey and Imberger (1991) related $\Gamma$ to two dimensionless parameters (overturn Froude number, $Fr_T = (L_R/L_C)^{2/3}$ and overturn Reynolds number, $Re_T = (L_C/L_K)^{4/3}$, where $L_R$ is Ozmidov scale, $L_C$ the scale of the most energetic overturn and $L_K$ the Kolmogorov scale), from laboratory experi-
ments. Imberger and Ivey (1991) classified observed turbulent events in this parameter space and inferred \( \Gamma \) by different generation processes of turbulence. Moum (1996), Ruddick et al. (1997) and St. Laurent and Schmitt (1999) also followed the definition by Oakey (1982; 1985) and obtained similar values of \( \Gamma \approx 0.2 \pm 0.1 \) in turbulent patches. Seim and Gregg (1994) observed the evolution and decay of Kelvin-Helmholtz (KH) instability in a tidal channel and found highly variable \( \Gamma \) (0.17-1.3) in each stage of instability and a higher average value of \( \Gamma = 0.58 \). Gargett and Moum (1995) and Seim and Gregg (1995) pointed out the difference in the definition of \( \Gamma \) in Seim and Gregg (1994) and \( \Gamma \) should be halved for comparison with past studies. Gargett and Moum (1995) also reported \( \Gamma \) is around 0.2 in a tidal front.

Laboratory experiments mainly investigate the ratio of potential energy gain to kinetic energy gain \( R_f \) in turbulent patches. Thorpe (1968) reproduced KH instability in tilted tube experiments, and Thorpe (1973) estimated \( R_f \) over one tilting experiment and obtained \( 0.21 < R_f < 0.27 \) (0.27 < \( \Gamma \) < 0.36). By using results from Thorpe (1973) and Koop (1976), Linden (1979) showed that \( R_f \) over one tilting experiment is a decreasing function of Richardson number. In theoretical studies, Thompson (1980), inspired by results of Thorpe (1973), showed that the maximum value of \( R_f \) in turbulence is equivalent to the critical Richardson number \( (R_{ic} = 0.25, \text{Miles 1961; Howard 1961}) \) from energy budgets. However, he also assumed that vertical eddy diffusivity is the same for momentum and density, that turbulence occurs in a limited region and that turbulence is always driven and has the same intensity when \( R_i < R_{ic} \). Winters et al. (1995) and Winters and D’Asaro (1996) further explored the energy balance in stratified turbulent events and obtained the exact expression for energy budgets. Since this framework is easy to reproduce in a direct numerical simulation (DNS), it becomes the basis for studies using the computational approach.

Due to recent advances in computer resources, DNS has been used for studying stratified
turbulence. Smyth (1999), Smyth and Moum (2000), Caulfield and Peltier (2000), Staquet (2000), Smyth et al. (2001) and Peltier and Caulfield (2003) reproduced KH instability in unforced shear flow and reported the time dependent behavior of efficiencies (their definitions of efficiencies are reviewed in Smyth et al. 2007). Caulfield and Peltier (2000) and Peltier and Caulfield (2003) introduced cumulative mixing efficiency and parameterized cumulative values during the turbulent stage as a function of Richardson number. Staquet (2000) and Smyth et al. (2001) also found the complex behavior of instantaneous ratio. Smyth et al. (2001) defined mixing efficiency averaged over one mixing event to take contributions other than turbulent stage into account, and examine dependences on Richardson number and Reynolds number. Smyth et al. (2007) and Carpenter et al. (2007) studied mixing efficiency in Holmboe waves. Smyth et al. (2007) suggested that net mixing efficiency is governed by the relative duration and intensity of the preturbulent and turbulent phases. These numerical studies used unforced shear as a background flow; however, turbulence induced by internal waves may be sensitive to the time evolution of the forcing fields. KH instability associated with internal-wave breaking was observed directly by Woods (1968). The phenomenon has been studied numerically (Bouruet-Aubertot and Thorpe 1999; Bouruet-Aubertot et al. 2001; Fringer and Street 2003), but there are still limitations of computer resources for reproducing turbulence due to wave-breaking and examining large parameter ranges.

The objective of this study is to increase our understanding of mixing efficiency in the ocean by using DNS of KH billows in a time-dependent mean flow designed to mimic a breaking internal wave. We focus on the relation between mixing efficiency and time-dependent forcing fields. We reproduce time-dependent forcing by solving the equations of motion in a tilted coordinate frame and allowing the tilt angle to vary in time. The resulting baroclinic torque generates a time-varying mean shear similar to what we would expect in a monochromatic internal wave. In this
framework, dimensional parameters that govern the flow include maximum buoyancy frequency $N$, forcing frequency $\omega$, maximum shear $\partial U/\partial z$, molecular viscosity $\nu$, molecular diffusivity $\kappa$ and layer thickness $H$. According to Buckingham’s $\Pi$ theorem (Kundu 1990), we can make four independent dimensionless parameters from these six dimensional numbers; Prandtl number $Pr = \nu/\kappa$, normalized forcing frequency $\omega/N$, Richardson number $Ri = N^2/(\partial U/\partial z)^2$ and Reynolds number $Re = H^2\nu\partial U/\partial z$. Since we explore turbulent mixing generated by KH instability over one wave cycle, first we evaluate mixing efficiency as a function of two dimensionless parameters which represent forcing wave properties: the minimum Richardson number achieved in one wave cycle, $Ri_{\text{min}}$, and the frequency of tilting compared with the buoyancy frequency, $\omega/N$. These choices allow us to investigate behavior of mixing efficiency in energetic and persistent mixing events inferred from the ocean internal wave spectrum (Garrett 2001).

In section 2, we explain our mathematical model and numerical solution methods. Section 3 reviews basic concepts for quantifying a mixing event. Section 4 describes the main results. We begin by examining the main effect of the periodic forcing, namely the deceleration of the means shear after instability growth has began. We then look at mixing statistics in various parameter ranges, including high-frequency waves similar to those observed by Woods (1968) and lower frequency waves that we expect to occur more commonly in the ocean. Dependences on Reynolds number and on the amplitude of the initial perturbation are also explored. In section 5, we compare DNS results with microstructure measurements via turbulent kinetic energy equation.

2. Computation method
a. Model equations in the tilted frame

The DNS model used in this study numerically integrates the incompressible equations of motion under the Boussinesq approximation. To represent time-dependent forcing, we write the equations in a coordinate system tilted at an angle \( \tau(t) \) from the horizontal,

\[
\begin{align*}
\frac{D u}{D t} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u - g \frac{\rho}{\rho_0} \sin \tau + 2 \frac{d \tau}{dt} w, \\
\frac{D v}{D t} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + \nu \nabla^2 v, \\
\frac{D w}{D t} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + \nu \nabla^2 w - g \frac{\rho}{\rho_0} \cos \tau - 2 \frac{d \tau}{dt} u,
\end{align*}
\]

where \( D/Dt = \partial/\partial t + u \cdot \nabla \), the velocity vector is defined \( \mathbf{u} = (u, v, w) \). The \( x \) and \( y \) axes are taken to be the streamwise and spanwise directions, respectively, while \( z \) is positive upward. The constant \( \rho_0 \) is the reference density, \( p \) is the pressure, \( \nu \) is the molecular viscosity and \( g \) is the gravitational acceleration. The tilt angle is a sinusoidal function of time

\[
\tau = a \sin \omega t,
\]

having the maximum amplitude \( a \) and varying with the frequency \( \omega \). The final terms in the \( u \) and \( w \) components of (1) are Coriolis accelerations arising from this time-dependent rotation. The incompressibility condition is expressed as

\[
\nabla \cdot \mathbf{u} = 0.
\]

An advection-diffusion equation is used to calculate the density \( \rho \):

\[
\frac{D \rho}{D t} = \frac{\nu}{Pr} \nabla^2 \rho,
\]

where \( \nu = 10^{-6} \text{ m}^2 \text{ sec}^{-1} \), \( Pr = 7 \). We use only one component for density rather than temperature and salt; therefore the possibilities of double diffusive processes and differential diffusion are
excluded (e.g. Ruddick et al. 1997). We impose a periodic condition for horizontal boundaries. No-flux and free slip conditions are imposed at \( z = \pm \frac{L_z}{2} \):

\[
w \mid_{z=\pm \frac{L_z}{2}} = 0, \quad \frac{\partial u}{\partial z} \mid_{z=\pm \frac{L_z}{2}} = 0, \quad \frac{\partial v}{\partial z} \mid_{z=\pm \frac{L_z}{2}} = 0, \quad \frac{\partial \rho}{\partial z} \mid_{z=\pm \frac{L_z}{2}} = 0. \tag{5}
\]

\( L_x \) is set twice longer than the wavelength of fastest-growing mode of KH billow estimated from the linear stability theory (e.g. Hazel 1972) to allow for pairing (Klaassen and Peltier 1989). Following Klaassen and Peltier (1991), we take the domain width \( L_y \) as larger than the half wavelength of one KH billow (quarter of \( L_x \)), and the domain height \( L_z \) is taken to be larger than the wavelength of one KH billow. We confirm that vertical boundaries do not suppress the pairing and breakdown of billows in all runs.

**b. Numerical methods**

Spatial discretization is via the Fourier pseudospectral method. Time stepping is accomplished using a third-order Adams-Bashforth method, except for viscosity and diffusion terms, which are evolved analytically. Grid spacing is isotropic and designed to resolve turbulence due to breakdown of KH billows (table 1). Further details of the DNS model are found in Smyth et al. (2005).

**c. Forcing and initial conditions**

At \( t = 0 \), the density anomaly has a two-layer form, with layers separated by a transition zone of thickness \( 2h_0 \)

\[
\rho = -\Delta \rho \tanh \frac{z}{h_0}, \tag{6}
\]
where $\Delta \rho$ is a density change across $h_0$. As the tilt angle $\tau$ departs from zero, gravitation acceleration generates a sheared velocity profile

$$u = \Delta u(t) \tanh \frac{z}{h_0},$$

(7)

where $\Delta u$ is the velocity change across $h_0$ and varies in time. As the shear increases, the minimum Richardson number drops from infinity, eventually becoming smaller than $1/4$, at which time the flow becomes unstable and the transition to turbulence begins.

As discussed in section 1, we evaluate mixing efficiency $\Gamma$ as a function of two dimensionless outer parameters; the projected minimum Richardson number $Ri_{\text{min}}$ and the frequency of tilting compared with the buoyancy frequency $\omega/N$ (table 1). In the absence of instability and neglecting molecular effects, $Ri$ can be related to the tilting angle $\tau$

$$Ri = \frac{N^2}{\left( \frac{\partial u}{\partial z} \right)^2} = \frac{\cos \tau}{N^2 \left( \int_0^t \sin \tau dt \right)^2}$$

(8)

(Thorpe 1971). The maximum tilt angle $\alpha$ in (2) is numerically estimated by setting $Ri = Ri_{\text{min}}$ when $\tau$ first returns to zero (figure 1).

Molecular effects on the mean flow cause the layer half-thickness to increase slowly from its initial value $h_0$. To minimize computational time and interface thickening, we start numerical runs when (8) first becomes $Ri = 0.25$ (figure 1). Initial conditions are derived from (6) and (7) with $\Delta u$ chosen to give $Ri = 0.25$. The Reynolds number is defined as,

$$Re_0 = \frac{h_0 \Delta u}{\nu},$$

(9)

For most of the experiments reported here, we choose $Re_0 = 300$. The effects of different $Re_0$ are discussed in section 4d. In some cases where the forcing timescale is very long, mixing is complete before $\omega t$ reaches the final value $2\pi$. For economy, these runs are terminated when
intensity of scalar mixing becomes molecular level. We use the reference buoyancy frequency $N_0$ for dimensionless time ($t_{nd} = N_0 t / (2\pi)$),

$$N_0 = Ri_{\text{min}}^{1/2} \frac{\Delta u}{h_0}.$$  
\hspace{1cm} (10)

We also add a disturbance to the initial profiles to efficiently reproduce the three-dimensional motion (Smyth et al. 2005). The perturbation consists of a random velocity field concentrated near $z = 0$ and having maximum amplitude equal to $b \times \Delta u$, in which $b$ is the amplitude (table 1). The effect of different amplitudes is discussed in section 4d.

Nondimensional domain lengths for streamwise, spanwise and vertical directions are $L_x / (2h_0) = 13.96$, $L_y / (2h_0) = 3.49$ and $L_z / (2h_0) = 6.98$, respectively. We anticipate that acceleration phases subsequent to $\omega t = 2\pi$ may drive additional mixing; however, by $\omega t = 2\pi$, the transition layer has invariably thickened so that billows emerging in later acceleration phases will have significantly increased wavelength. As a result, the domain dimension chosen to accommodate the initial instability will no longer be appropriate. For this reason, we terminate all simulations no later than $\omega t = 2\pi$.

3. Mixing diagnostics

The volume averaged kinetic energy $\mathcal{K}$ and potential energy $\mathcal{P}$ are defined,

$$\mathcal{K} = \frac{1}{2} < u \cdot u >_V, \quad \mathcal{P} = \frac{g}{\rho_0} < \rho z >_V,$$  
\hspace{1cm} (11)

where $< >_V$ is average over computation domain. Following Winters et al. (1995), the volume-averaged energy budget for stratified turbulence can be written,

$$\frac{d\mathcal{K}}{dt} = -\mathcal{B} - \varepsilon,$$  
\hspace{1cm} (12)
\[ \frac{dP}{dt} = B + \Phi. \] (13)

\( B \) is the buoyancy flux and \( \varepsilon \) the kinetic energy dissipation rate. \( \Phi \) is the irreversible changing rate of \( P \) due to molecular diffusion of the instantaneous linear density profile,

\[ \Phi = \frac{\nu g \cos \tau \Delta \rho}{Pr \rho_0 L_z}. \] (14)

We also subdivide the potential energy into available potential energy \( P_a \) and background potential energy \( P_b \) as,

\[ P = P_a + P_b, \] (15)

where,

\[ P_b = \frac{g}{\rho_0} < \rho_b z > V. \] (16)

\( \rho_b \) represents the background density profile obtained by three-dimensional reordered density and the minimum potential energy state; therefore its change is due to irreversible mixing process (Winters et al. 1995). \( P_a \) becomes,

\[ P_a = P - P_b. \] (17)

The changing rate of the potential energy can be subdivide into two components and the background density profile is related to irreversible mixing process (Winters et al. 1995),

\[ \frac{dP_a}{dt} = B - M, \] (18)

\[ \frac{dP_b}{dt} = M + \Phi. \] (19)

\( P_a \) exchanges energy with \( K \) via \( B \). \( M \) is the changing rate of the background density due to diapycnal mixing,

\[ M = \frac{dP_b}{dt} - \Phi = \frac{g}{\rho_0} \frac{d < \rho_b z > V}{dt} - \Phi. \] (20)
The instantaneous ratio of diapycnal flux to $\varepsilon'$ is defined (Carpenter et al. 2007; Smyth et al. 2007),

$$\Gamma_i = \frac{M}{\varepsilon'}.$$  \hfill (21)

Here, $\varepsilon'$ is the dissipation rate due to velocity fluctuations, defined by $u' = u - \bar{u}$ and does not correspond to dissipation in Winters et al. (1995) (see also Gill 1982). Since high $\Gamma_i$ does not necessarily mean large flux or intense mixing, we also use normalized $M (\mathcal{M}n_i)$ and $\varepsilon' (Re_i)$ for comparison,

$$\mathcal{M}n_i = \frac{M}{\Phi}, \quad Re_i = \frac{\varepsilon'}{Pr \Phi}.$$  \hfill (22)

These expressions for the diagnostics are only strictly correct at times when the tilt angle is zero; however, we evaluate it continuously in time to facilitate comparison with previous studies. Ambiguities of diagnostics in the tilting frame are discussed in Appendix.

Net mixing parameters for each run are computed at $\omega t = 2\pi$. These parameters can be defined over one wave cycle,

$$Re_c = \frac{\int_{t_1}^{t_2} \varepsilon' dt}{Pr \int_{t_1}^{t_2} \Phi dt}, \quad \mathcal{M}n_c = \frac{\int_{t_1}^{t_2} M dt}{\int_{t_1}^{t_2} \Phi dt}, \quad \Gamma_c = \frac{\int_{t_1}^{t_2} M dt}{\int_{t_1}^{t_2} \varepsilon' dt},$$  \hfill (23)

where $t_1$ is defined as the time when first the vertical eddy diffusivity overcomes the molecular diffusivity $\mathcal{M}n_i > 1$ and $t_2 = 2\pi/\omega$. If the forcing frequency is small, mixing decreases to the same magnitude as that due to molecular diffusion before one wave cycle is achieved. In those cases, we define $t_2$ the time just before $\mathcal{M}n_i < 1$. We mainly examine these time averaged values because this allows us to ignore time dependence and may also correspond to the average of many samples randomly observed in the ocean.
4. Results

Here, we explain how $\Gamma_c$ depends on forcing fields. First, we connect our work with previous studies by showing how mixing is suppressed in the deceleration phase. Second, we investigate dependences of $\Gamma_c$ on $Ri_{min}$ and $\omega/N$ with fixed $Re_0$ and $b$. Third, we explain further dependences of $\Gamma_c$ on $Re_0$ and $b$. Last, we focus on low-frequency forcing fields with higher $Re_0$.

a. Effects of mean flow deceleration

The present experiments differ from previous simulations of the unforced case (e.g. Smyth et al. 2001) mainly because the mean shear that drives instability exists only for a limited time before it is decelerated by the external forcing. The mean flow is accelerated from the beginning of the run until $\omega t = \pi$, and then decelerated from $\omega t = \pi$ to $\omega t = 2\pi$. In the acceleration phase, KH billows grow much as they do in the unforced case. The subsequent deceleration, however, can have dramatic effects on disturbance evolution. We examine the effect of the deceleration phase by comparing runs with and without deceleration. To omit deceleration, we simply leave the tilt angle ($\alpha$) as zero after $\omega t = \pi$.

Figure 2 shows results for the case with $Ri_{min} = 0.08$ and $\omega/N = 0.05$ (AD7 and A1 in table 1). After $Ri$ drops below 1/4, a train of KH billows emerges. The computational domain accommodates two such billows as shown in figure 2a. The billows pair (figure 2b), then break and become turbulent (figures 2c and 2d). Mean flow deceleration begins during the pairing process (between figures 2a and 2b). The effect of deceleration is clear in figure 2c. With no deceleration phase (A1, lower frames in figure 2), the paired KH billow is sheared into two parts by the strong mean flow. With deceleration (AD7), the billow relaxes to a more circular shape. As the mean flow
decelerates toward zero, the billow collapses into a field of weakly turbulent gravity waves (figure 2d, upper), whereas without deceleration (A1), the mean shear generates sustained turbulence (figure 2d, lower).

Energy budgets can quantitatively explain these mixing events. KH instability occurs after the acceleration (figure 3a), thus the maximum potential energy is diminished between \( t_{nd} = 12 \) and 13 (figure 3b). The subsequent turbulence \( Re_i \) after \( t_{nd} = 13 \) is also diminished as well as the diapycnal flux \( Mn_i \) (figures 3c and 3d). AD7 has maximum values of \( Re_i \) and \( Mn_i \) after breakdown at \( t_{nd} = 13.4 \). However, there is time lag in A1 (\( t_{nd} = 14 \) in figure 3c and \( t_{nd} = 13.7 \) in figure 3d) because the secondary instability of KH billows (Klaassen and Peltier 1985) makes maximum \( Mn_i \) during breakdown. The difference of \( \Gamma_i \) seems small in figure 3e. Before breakdown (\( t_{nd} < 12 \)), \( \Gamma_i \approx 0.7 \) corresponds to highly efficient preturbulent mixing (Smyth et al. 2001). \( \Gamma_i \approx 0.2 \) is achieved in turbulence stage and mixing in A1 lasts longer. Contributions to diapycnal flux from the preturbulent phase are more important in AD7 because deceleration suppresses turbulence at later times. Cumulative values have differences due to different intensity and duration of the preturbulent and turbulent phases; \( Re_c = 4.98 \) (11.43), \( Mn_c = 9.18 \) (18.14) and \( \Gamma_c = 0.26 \) (0.23) for AD7 (A1). \( Re_c \) in AD7 are less than the half value of that in A1 suggesting strong modification due to a deceleration phase.

If \( Ri_{min} \) is larger, KH billows will grow slower (e.g. Hazel 1972) and be suppressed by mean flow deceleration. If \( \omega / N \) is larger, the faster arrival of the deceleration will suppress the growth of KH billows. Therefore, the relative timing of billow growth and mean flow deceleration will be important for a mixing event.
b. Forcing by high-frequency internal waves

In the classic observations of Woods (1968), KH billows were generated by gravity waves propagating on thin layers of strong stratification and were related to existing theories of shear instability (Miles and Howard 1964). The observed gravity waves had wavelength $\lambda \approx 10$ m and the thickness of the interface was $h \approx 0.1$ m. Using the dispersion relation for interfacial waves (e.g. Kundu 1990), the normalized frequency of the wave can be estimated,

$$\frac{\omega}{N} = 0.5 \times k h \approx O(0.1),$$

where $k$ is the wave number and the reduced gravity is approximated by $g' \approx h \times N^2$. Here, we choose $\omega/N = 0.1$ and $Ri_{\text{min}} = 0.07$ (AD6) and 0.08 (AD13) corresponding to the lower limit of Richardson number in Woods (1968) (figure 4 and table 1). When $Ri_{\text{min}}$ is larger than this, there is no full development of KH billows due to the slower growth of billows. $M n_c$ becomes close to 1 (molecular value); thus mixing due to shear instability is negligible (figure 5).

In both cases, the deceleration phase of the high-frequency wave strongly affects the initial growth of KH billows (figure 5) and there is no pairing. Mixing is not completed within one wave cycle. $Re_c$ and $M n_c$ are small (figure 4), but $\Gamma_c$ is higher ($\geq 0.3$) because the efficient preturbulent mixing (Smyth et al. 2001) contributes significantly to the net diapycnal flux. At low $Ri_{\text{min}}$ (case AD6), preturbulent mixing is intense and $\Gamma_c$ is higher (figure 5).

c. Effects of decreased forcing frequency

The assumption that the maximum tilt angle $a$ is small is necessary for some elements of our analysis, and is also well justified by the fact that even strongly nonlinear interfacial waves in the
ocean usually have steepness much less than unity (e.g. Moum et al. 2003). Assuming $a \ll 1$ (in radian angle), (2) and (8) may be combined to yield

$$Ri_{\text{min}} \approx \frac{\omega^2}{4a^2N^2},$$

so that $a \ll 1$ implies

$$\frac{\omega}{N} \ll 2\sqrt{Ri_{\text{min}}}. \quad (26)$$

Combining this with the requirement for instability, $Ri_{\text{min}} < 1/4$, we find that the assumption of small tilt angle is equivalent to $\omega/N \ll 1$. The condition of low steepness, combined with the condition for instability, is therefore equivalent to a requirement that the wave frequency be much smaller than the buoyancy frequency. The observed waves discussed in the previous subsection had $\omega/N \sim O(0.1)$, and therefore satisfied this criterion marginally. In the remainder of this study, we will focus on cases with $\omega/N < 0.1$.

In our parameter space (figure 4), smaller $Ri_{\text{min}}$ or smaller $\omega/N$ tends to make larger $Re_c$ and $Mn_c$ because of the faster growth of KH billows or slower arrival of deceleration. When $Ri_{\text{min}}$ is larger and $\omega/N$ is smaller, it takes long time before the onset and an interface is diffused via molecular process. The thickness of the velocity interface becomes greater than that of the density interface due to different diffusivities; therefore the projected $Ri_{\text{min}}$ from (8) will not work well and Holmboe instability (Holmboe 1962) may occur. This is most likely in low $Re_0$ cases (figure 5).

Over the range of parameter values shown in figure 4, $\Gamma_c$ remains within about 20% of 0.3. $\Gamma_c$ is higher when preturbulent mixing is important and turbulence is suppressed by the deceleration phase as mentioned in section 4b. When $Ri_{\text{min}}$ is smaller, this higher $\Gamma_c$ appears at larger $\omega/N$ because $\Gamma_c$ is controlled by the relative timing of two events; the growth of KH billows and arrival
of the deceleration phase. The boundary between the pairing and no pairing cases seems to be defined by this relative timing (figure 4). For $Ri_{\text{min}} = 0.07$ and $\omega/N = 0.07$ (AD3 in table 1), KH billows can grow faster and pair, then deceleration suppresses turbulence and makes $\Gamma_c$ higher.

If both $Ri_{\text{min}}$ and $\omega/N$ are small, we could expect that a mixing event may be completed within the acceleration phase (figure 5). We further explore this situation with different initial conditions and will show that $\Gamma_c \approx 0.2$. In preparation for this, we first examine effects of different projected Reynolds number $Re_0$ and the amplitude of the initial perturbation $b$, showing that higher $Re_0$ and $b$ intensify the development of KH billows and allow us to avoid the interface thickening in the low frequency forcing.

d. Sensitivity to $Re_0$ and $b$

In addition to $Ri_{\text{min}}$ and $\omega/N$, $\Gamma_c$ depends on projected Reynolds number $Re_0$ and the amplitude of the initial perturbation $b$. Furthermore, the fact that $Re_0$ used in our DNS is smaller than typical oceanic values gives some doubt to generality of our results for the ocean. To examine sensitivity to these additional parameters, we add DNS runs with different values of the projected $Re_0 = 500$ (ADR1) and $Re_0 = 800$ (ADR2) and different $b$ (AD- with $b = 0.05$ and AD+ with $b = 0.2$) in table 1. We use $Ri_{\text{min}} = 0.08$ and $\omega/N = 0.07$, where the deceleration has a significant effect on mixing at $Re_0 = 300$ (figure 4 and AD10 in table 1).

As $Re_0$ becomes higher, KH billows grow faster and pairing can occur. Higher $Re_0$ can make the growth of the secondary instability faster (Klaassen and Peltier 1985); therefore, efficient mixing during pairing and breakdown is enhanced and stronger than in $Re_0 = 300$ runs (table 1). ADR2 has the larger $\Gamma_c$ (table 1) because the deceleration suppresses mixing just after breakdown.
Although mixing continues after one wave cycle in higher $Re_0$ runs, the tilting is stopped after one wave cycle.

For AD-, the smaller initial disturbance amplitude $b$ makes the onset slower, thus mixing is suppressed by deceleration at an earlier phase in billow evolution. For AD+, the larger $b$ makes the onset faster than AD10 and promotes pairing and subsequent intense, and less efficient, turbulence. Thus the effects of preturbulent mixing become less. This reduced importance of preturbulent mixing is also found in Holmboe wave simulations (Smyth et al. 2007). The result is that $\Gamma_c$ becomes smaller and intensity of mixing is similar to AD7.

Next, we conduct the higher $Re_0 = 500$, larger $b = 0.2$ and longer duration of forcing ($\omega/N = 0.05$) runs (with deceleration, ADR+, and without deceleration, AR+, in table 1) to see whether the effect of deceleration is reduced. KH billows start to develop at $t_{nd} = 9.5$. There is a little difference of maximum potential energy in developing stage because billows can grow before the deceleration affects on mixing (figure 6b). $Re_i$ is high during and after breakdown because of the faster growth of the secondary instability ($12 < t_{nd} < 13$ in figure 6c). $Mn_i$ becomes high around breakdown because of higher $Re_i$ and $\Gamma_i \approx 0.4$ ($12 < t_{nd} < 13$ in figures 6d and 6e). The highest $\Gamma_i$ in the initial roll-up of KH billows does not contribute much to the diapycnal flux. However, the efficient mixing around breakdown is more important in the higher $Re_0$ case. In the decaying turbulence stage, $\Gamma_i$ decreases to 0.2 (figure 6e). The effect of deceleration appears after $t_{nd} = 12$ but seems less important than that shown in figure 3 (AD7 and A1) and mixing in ADR+ lasts longer than AD7. $Re_e, Mn_e$ and $\Gamma_c$ become 18.20 (25.20), 42.92 (55.18) and 0.34 (0.31) for ADR+ (AR+) (table 1). $\Gamma_c$ is higher than 0.3 due to efficient mixing around breakdown and ADR+ has higher $\Gamma_c$ because of the deceleration. Since differences of cumulative values between these runs are smaller than those discussed in section 4a, the deceleration phase appears to become...
less important if $\omega/N$ is smaller and $Re_0$ is higher. $\Gamma_c$ in ADR+ increase 30% larger than AD7 showing dependence on $Re_0$ because mixing during breakdown is more efficient in higher $Re_0$.

e. Forcing by highly unstable, low-frequency internal waves

When both $Ri_{\text{min}}$ and $\omega/N$ are small, mixing by fully-developed turbulence is not suppressed by wave deceleration, and we therefore expect to recover the canonical turbulent mixing efficiency $\Gamma_c = 0.2$. To test this speculation, we examine a case with low $Ri_{\text{min}} = 0.03$ and low $\omega/N = 0.025$. We also use increased values of $Re_0$ and $b$ to promote rapid onset of turbulence (ADRL+ in table 1).

KH billows start to develop and break before $Ri_{\text{min}}$ is achieved ($t_{nd} < 15$ figures 7a and 7b). Intense mixing is almost completed before deceleration begins at $t_{nd} = 20$ (figures 7c and 7d). Since the onset occurs in acceleration, the maximum potential energy is larger than before ($t_{nd} = 14.2$ in figure 7b). Turbulence becomes intense after breakdown and the maximum value is achieved after breakdown ($t_{nd} > 16$ in figure 7c). However, the larger $Mn_i$ (figure 7d) is achieved by two reasons; one is the high diapycnal flux during breakdown with higher $\Gamma_i (> 0.4)$ (figure 7e) as before. The other is flux after breakdown with larger $Re_i$ with $\Gamma_i \approx 0.2$ (figure 7e) due to turbulence. Since the forcing keeps accelerating the mean shear, less efficient mixing ($\Gamma_i = 0.2$) continues longer until the end of the event and cancels higher $\Gamma_i$ around breakdown. We note that the acceleration during breakdown makes mixing more efficient ($\Gamma_i$ up to 0.6 at $t_{nd} = 15$) than that during pairing ($t_{nd} = 13$) in figure 7e. $\Gamma_c$ is near 0.2 even though the efficient mixing is reproduced in this higher $Re_0$ run (table 1).

From above results, we may infer ocean mixing due to a low-frequency wave with the high
Reynolds number as follows. The high Reynolds number can make onset and subsequent transition to turbulence faster. It makes efficient mixing around breakdown, but this faster transition also makes duration of turbulence longer and cancels the efficient mixing. Finally, mixing efficiency becomes $\approx 0.2$. The lack of this long duration of turbulence may explain why laboratory experiments (Thorpe 1973) and DNS (e.g. Smyth et al. 2001) had higher mixing efficiency ($\Gamma_c \geq 0.3$). We may need to be careful not to assume 0.2 for the ocean mixing forced by high-frequency internal waves, because in that case a large fraction of the mixing is accomplished in the highly-efficient preturbulent phase.

We note that, in the longer forcing run, there could be an additional pairing if our boundary conditions allowed it, and that could alter the value of $\Gamma_c$. To simulate this would require more memory than is available. Instead, we address the issue indirectly by examining a case in which the first pairing is suppressed. An auxiliary simulation was conducted with $Ri_{\text{min}} = 0.08$, $\omega/N = 0.05$ and $Re_0 = 300$, similar to AD7, but with $L_e$ halved so as to suppress the paring (AD7x in Table 1). The result was that $\Gamma_c$ was increased from 0.26 (AD7) to 0.32 (AD7x). This indicates that suppression of pairing results in larger $\Gamma_c$. Therefore, if pairing was allowed in ADRL+, we expect that $\Gamma_c$ would be smaller.

We add two runs to see effect of forcing parameters on above results. One run has the same parameters except the smaller $Re_0 (= 325)$ (ADL+ in Table 1). The another run has an intermediate $Ri_{\text{min}} (= 0.05)$ and $\omega/N (= 0.035)$ with the same $Re_0 (= 500)$ (ADRI+ in Table 1) to see effects of $Ri_{\text{min}}$ and $\omega/N$. In ADL+, the pairing instability is not reproduced and one KH billow is absorbed in the interface (draining instability in Klaassen and Peltier 1989), but the forcing field accelerate the mean shear during turbulent phase. $Mn_c$ is smaller than ADRL+ because mixing during breakdown is less efficient in lower $Re_0$ (Table 1). In these long forcing runs, the effect of
$Re_0$ on $\Gamma_c$ is small. In ADRI+, the forcing field keeps accelerating KH billows during breakdown, but the deceleration phase is arrived faster than $\omega/N = 0.025$ cases. Therefore, the duration of turbulence is shorter and there is a suppression of turbulence. This run has the higher $Re_c$ and $Ma_c$ than ADRL+ maybe because the lower $Ri$ is achieved during KH instability due to the higher forcing frequency and the non-steadiness of the forcing field becomes important (table 1). However, $\Gamma_c$ is higher (0.29) than ADRL+ and ADL+ as expected.

5. Comparison with observations

We have shown that mixing efficiency depends on forcing, but it is not clear yet whether we can use $\Gamma = 0.2$ in the ocean mixing. Here, we compare DNS results with microstructure measurements by using turbulent kinetic energy (TKE) equation averaged over one mixing event. After averaging over an event, we can write the TKE equation following (12) and Smyth et al. (2007),

$$S = B + \varepsilon',$$

(27)

where $\bar{}$ is average over one event via the same way as (23). $S$ is the shear production term. By using the time-averaged flux Richardson number, $[R_f] = [B]/[S] = \Gamma_c/(1 + \Gamma_c)$, where $[B] = [M]$, we can rewrite (27) as

$$[B] = \Gamma_c[\varepsilon'].$$

(28)

Then we introduce two measurable length scales (Dillon 1982; Garrett 2001), the Thorpe scale ($L_T$) and the Ozmidov scale ($L_O$), which may be related to KH instability and subsequent mixing in the time averaged sense. $L_T$ is obtained from the kinematical definition (Thorpe 1977) to estimate the typical parcel vertical displacement and defined as the mean square root of the Thorpe
displacement, $\delta_T$, usually in the mixing (or transition) layer,

$$L_T = \langle \delta_T^2 \rangle_{\text{mix}}^{1/2},$$

where $\langle \cdot \rangle_{\text{mix}}$ is volume average over mixing layer. $L_O$ is the length scale at which turbulence is strongly affected by buoyancy and is defined as (Ozmidov 1965)

$$L_O = \left( \frac{\langle \varepsilon' \rangle_{\text{mix}}}{\langle N^3_s \rangle_{z_{\text{mix}}}} \right)^{1/2},$$

where $\langle N^3_s \rangle_{z_{\text{mix}}}$ is vertical average of buoyancy frequency estimated from three-dimensional reordered density field within the mixing layer

$$N_s^2 = -\frac{g \cos \tau}{\rho_0} \frac{d\rho_b}{dz}.$$  

In our model, mixing is confined within the transition layer, thus we define the vertical length ($z_{\text{mix}}$),

$$z_{\text{mix}}(t) = \max |z_1(x, y, t) - z_2(x, y, t)|,$$

where $z_1$ and $z_2$ satisfy $\rho(x, y, z_1, t) = -\Delta \rho \tanh(1)$ and $\rho(x, y, z_2, t) = \Delta \rho \tanh(1)$ (Smyth et al. 2005), respectively. If we further introduce the buoyancy length scale,

$$L_B = \left( \frac{\langle B \rangle_{\text{mix}}}{\langle N^3_s \rangle_{z_{\text{mix}}}} \right)^{1/2},$$

(27) becomes

$$[R_{BT}]^2 = \Gamma_c [R_{OT}]^2,$$

where $R_{BT}$ and $R_{OT}$ are ratio of $L_B$ and $L_O$ to $L_T$, respectively. (34) suggests that, in the time average sense, TKE equation can be described with three dimensionless parameters ($R_{BT}$, $\Gamma_c$ and $R_{OT}$). $R_{OT}$ is also related to estimation of $\varepsilon$ from reordered density profiles (e.g. Galbraith and
Kelley 1996) and is observed as 0.79 (Dillon 1982) and 0.66 (Crawford 1986). Garrett (2001) proposed $R_{BT}^2 \approx 0.1$. Although these variables are defined in the mixing layer for correspondence with observations, we use $\Gamma_c$ obtained in the whole domain. All variables need to be averaged over one mixing event, but our DNS is limited to one wave cycle. The fact that $z_{mix}$ is a function of time may violate the mathematical correctness of energy budget.

\[ \frac{[L_O]}{h_0} \text{ vs } \frac{[L_T]}{h_0} \text{ diagram (figure 8a) describes how mixing is modified by forcing. When the roll-up of KH billows is suppressed, both } \frac{[L_T]}{h_0} \text{ and } \frac{[L_O]}{h_0} \text{ are small. When the pairing instability occurs, both } \frac{[L_T]}{h_0} \text{ and } \frac{[L_O]}{h_0} \text{ are larger than no-pairing cases. However, for some pairing cases, mixing is not completed within one wave cycle, thus } \frac{[L_T]}{h_0} \text{ is large, but } \frac{[L_O]}{h_0} \text{ is small. } R_{OT} \text{ becomes smaller than no-pairing cases. For the long forcing runs, } \frac{[L_O]}{h_0} \text{ can have larger values within one wave cycle, then } R_{OT} \text{ is comparable to in-situ observations (Dillon 1982; Crawford 1986). } \Gamma_c \text{ vs } R_{OT} \text{ diagram (figure 8b) also shows dependences on forcing fields in terms of energy budget as (34). When the deceleration phase suppresses a mixing event, } R_{OT} \text{ is small and } \Gamma_c \text{ is high. } R_{BT}^2 \text{ is around 0.07. If deceleration does not suppress turbulence until late in the event, } R_{OT} \text{ is higher, } \Gamma_c \text{ is smaller and } R_{BT}^2 \text{ is around 0.15. }

These results suggest the importance of the acceleration in turbulence stage to reproduce observed values in DNS. Since this scaling shows clear dependences on forcing fields, application of this scaling to in-situ microstructure measurements will be useful for further understanding of mixing efficiency. The in-situ observation could give us some idea for high Reynolds number turbulence which we cannot explore in present computers. We also need to compare the statistics of turbulent events and effects of different sampling methods in the ocean and DNS (e.g. Wijesekera and Dillon 1997; Smyth et al. 2001).
6. Summary

We investigated the dependence of mixing efficiency on forcing fields via direct numerical simulation (DNS) of Kelvin-Helmholtz (KH) instability. We used a tilted coordinate system to reproduce time varying forcing fields which mimic a single internal wave, so that the mean shear that drives instability exists only for a limited time before it is decelerated. The cumulative mixing efficiency $\Gamma_c$ is defined as buoyancy flux as a fraction of the energy dissipation rate, both averaged over one wave cycle. We described $\Gamma_c$ as a function of the projected minimum Richardson number $Ri_{\text{min}}$, normalized forcing frequency $\omega/N$, the projected Reynolds number $Re_0$ and amplitude of the initial perturbation $b$. We showed sensitivities of $\Gamma_c$ on these parameters as well as the detail of one mixing event. We also found a limiting case in which the canonical value of mixing efficiency 0.2 is recovered. The time averaged Thorpe scale and Ozmidov scale are derived from turbulent kinetic energy equation following Dillon (1982) and Garrett (2001) and compared with past microstructure measurements.

A mixing event that results from shear instability mixes via two distinct mechanisms which occur in sequence. The first is preturbulent mixing, in which a growing, wavelike disturbance exerts a persistent, compressive strain on scalar gradients, sharpening the gradients and thereby accelerating diapycnal flux. This mixing occurs with relatively little dissipation of kinetic energy and is in this sense highly efficient. The second mechanism is turbulence, which develops as the instability grows beyond some amplitude threshold. In contrast to preturbulent mixing, turbulence exerts a strain field that varies rapidly in time. As a result, scalar gradients do not have time to adjust to the optimal orientation for compression (Smyth 1999). This rapidly fluctuating turbulent strain is also effective at dissipating kinetic energy, with the net result that instantaneous mixing
efficiency is reduced to somewhere near the canonical value 0.2. In a forced mixing event, forcing governs mixing efficiency by altering the relative duration and intensity of the preturbulent and turbulent phases. Detailed results may be summarized as follows:

- When $Re_0$ and $b$ are fixed, $\Gamma_c$ varies weakly as a function of $Ri_{min}$ and $\omega/N$. In this parameter space, the relation between the growth rate of KH billows and arrival of deceleration phase controls mixing. When $Ri_{min}$ and $\omega/N$ are high, there is no mixing because the deceleration phase suppresses mixing. If $Ri_{min}$ or $\omega/N$ is sufficiently reduced, KH billows can grow, pair and break even though the deceleration suppresses the latter part of the mixing event. $\Gamma_c$ is around 0.3 because highly efficient preturbulent mixing (Smyth et al. 2001) dominates diapycnal flux and less efficient turbulence after breakdown is suppressed by the deceleration.

- Higher $Re_0$ and larger $b$ act to accelerate the initial growth and secondary instability of KH billows (e.g. Smyth et al. 2007). KH billows can grow before the deceleration phase arrives, thus changing intensity of mixing as well as $\Gamma_c$. When both $Re_0$ and $b$ are higher and $\omega/N$ is smaller, the effects of a deceleration become less important because of intense turbulence. However, if the forcing field does not accelerate mean flow long enough, $\Gamma_c$ in higher $Re_0$ runs is larger than 0.3 because of efficient mixing late in the preturbulent phase.

- When $Ri_{min}$ and $\omega/N$ become small, KH billows onset before $Ri_{min}$ is achieved. Then the intense but less efficient turbulence acts over a long time before deceleration begins and therefore contributes more to the net mixing than the highly efficient preturbulent mixing; thus $\Gamma_c$ is near the canonical value of 0.2 even in the run with the higher $Re_0$.

- Following the length scale arguments by Dillon (1982) and Garrett (2001), the turbulent
kinetic energy equation can be written in terms of three nondimensional numbers; $\Gamma_c$, $R_{OT}$ (the ratio of Ozmidov scale $L_O$ to Thorpe scale $L_T$), and $R_{BT}$ (the ratio of Buoyancy scale $L_B$ to Thorpe scale) with average over one event. Both $L_O$ vs $L_T$ and $\Gamma_c$ vs $R_{OT}$ diagrams can capture the transition from suppression to acceleration of turbulent phase by forcing fields. Those numbers are close to the past observed values in cases where turbulence is able to develop and persist before mean flow deceleration.

Our study cannot investigate all parameter ranges because of the limitation of computer resources. Effects of the high Reynolds number and multiple pairing of billows should be examined further. The scaling of $\Gamma$ by $R_{OT}$ can be studied with in-situ microstructure observation. Effects of subsequent waves need to be studied in high-frequency wave cases. Reproducing internal-wave breaking via wave-wave interaction in DNS will give more information about $\Gamma$ in context of the ocean mixing. KH instability in non-steady forcing fields itself deserves further study.

Appendix

Errors in energy diagnostics

The energy diagnostics and three dimensional density reordering computation used in this study are only strictly correct when tilting angle is 0. Thus we assess the error in diagnostics by comparing $\Gamma_c$ from density reordering method with the cumulative mixing efficiency $\Gamma_c$ from differences in potential energies at $\omega t = \pi$ and $\omega t = 2\pi$ (end of one wave cycle). We use run AD7 (figure 3) for comparison; thus we can also assess the effect of termination before one wave cycle is completed. The time series of $\Gamma_c$ at each time step ($t_2 = t$ in (23)) and $\Gamma_c$ from the potential energy
difference show that there is almost no difference at $\omega t = 2\pi \ (t_{nd} = 20$ in figure 9). $\Gamma_c$ is 0.262 and that from the potential energy difference is 0.265 (about 1% difference). We note that there is also almost no difference of $\Gamma_c$ at $t_{nd} = 18.05 \ (Mn_i = 1$ is achieved at end of mixing; $\Gamma_c = 0.263$) and $\Gamma_c$ at $\omega t = 2\pi$ (less than 1% difference). The same analysis with other DNS runs suggests that errors in diagnostics are within a few percent and our results are robust.

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Table 1. Parameters and results for numerical simulations. $Ri_{\text{min}}$ is projected minimum Richardson number. $\omega/N$ is normalized forcing frequency. $a$ is the maximum tilt angle (deg). $b$ is the amplitude factor for the initial perturbation. $N_x$, $N_y$ and $N_z$ are coarse components grid numbers. Grid numbers are doubled for a scalar variable. $Re_c$ is the cumulative Reynolds number, $Mn_c$ is the cumulative normalized diapycnal flux, and $\Gamma_c$ is the cumulative mixing efficiency. $[R_{OT}]$ is the ratio of time averaged Ozmidov scale to Thorpe scale. $[R_{BT}]^2$ is the squared ratio of time averaged Buoyancy length scale to Thorpe scale.

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<th>$a$</th>
<th>$b$</th>
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<th>$Mn_c$</th>
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Figure Captions

FIG. 1. Tilting angle and projected Richardson number. (a) tilting angle (b) projected Richardson number. $t_{int}$ is the time when a simulation is started ($Ri = 0.25$) and $t_{min}$ the time when $Ri_{min}$ achieved. Horizontal axis is dimensionless time and $N_0$ is defined in (10).

FIG. 2. Density field for KH billows forced with (upper frames; AD7) and without (lower frames; A1) a deceleration phase. Colored values cover the middle 3/5 of the density range; higher and lower values are transparent. Actual domain length in computation is used for axes. (a) $t_{nd} = 11.1$ (b) $t_{nd} = 12.8$ (c) $t_{nd} = 14.4$ (d) $t_{nd} = 16.3$.

FIG. 3. Time series of a DNS run ($Ri_{min} = 0.08$ and $\omega/N = 0.05$). Solid line shows the run with deceleration phase (AD7) and dashed line that without deceleration (A1). Horizontal axis is dimensionless time. (a) tilting angle (b) total (thin line) and background (thick line) potential energies normalized by $P_0 = g/\rho_0 < \bar{\rho}z >_z$ (c) instantaneous dimensionless kinetic energy dissipation rate $Re_i$ (d) instantaneous dimensionless diapycnal buoyancy flux $Mn_i$ (e) instantaneous ratio of diapycnal buoyancy flux to kinetic energy dissipation rate $\Gamma_i$.

FIG. 4. Results in $Ri_{min}$ and $\omega/N$ spaces for $Re_0 = 300$. Numbers in columns show cumulative buoyancy Reynolds number $Re_c$, cumulative dimensionless diapycnal flux due to mixing $Mn_c$ and cumulative mixing efficiency $\Gamma_c$. Pairing is observed at the open circle and not at the cross.

FIG. 5. Schematic in $Ri_{min}$ and $\omega/N$ space.

FIG. 6. Same as figure 3 but for $Ri_{min} = 0.08$ and $\omega/N = 0.05$ with higher $Re_0 = 500$ and $b = 0.2$. Solid line shows the run with deceleration phase and dashed line that without deceleration.

FIG. 7. Same as figure 3 but for $Ri_{min} = 0.03$ and $\omega/N = 0.025$ with $Re_0 = 500$ and $b = 0.2$. Only the case with deceleration phase is shown. The deceleration starts at $N_0 t/(2\pi) = 20$.

FIG. 8. (a) $[L_O]/h_0$ vs $[L_T]/h_0$ diagram. (b) $\Gamma_c$ vs $[R_{OT}]$ diagram. Runs are with the deceleration phase are plotted. Open circle is $Re_0 = 300$ the pairing is observed (AD1-AD4, AD7-AD9 and AD+). Cross is the same as open circle but no pairing (AD5-AD6, AD10-AD16 and AD-). Triangle is $Re_0 = 500$ or 800 (ADR1, ADR2 and ADR+). Diamond is the long forcing cases (ADR1+, ADL+ and ADRI+). Dotted lines are $R_{OT} = 0.79$ (Dillon 1982) and $R_{OT} = 0.66$.
(Crawford 1986). Solid lines in (a) are $R_{OT} = 0.5$ and $R_{OT} = 1.0$. Dashed line in (b) is $R^{2}_{BT} = 0.1$ by Garrett (2001).

**Fig. 9.** Time series of $\Gamma_{e} (Ri_{min} = 0.08$ and $\omega/N = 0.05)$. Horizontal axis is dimensionless time. Thick dotted line is $\Gamma_{e}$ from the potential energy difference between $\omega t = \pi$ and $\omega t = 2\pi$. Thick dashed line is time when $Mn_i = 1$ is achieved.
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Mixing affected by deceleration $\Gamma_c \geq 0.3$

Transient $\Gamma_c \approx 0.3$

Mixing in acceleration $\Gamma_c \approx 0.2$

No mixing due to shear instability

Long forcing effects (when $Re_0$ is small)

Fig. 5. Schematic in $Ri_{min}$ and $\omega/N$ space.
Fig. 6. Same as figure 3 but for $Ri_{min} = 0.08$ and $\omega/N = 0.05$ with higher $Re_0 = 500$ and $b = 0.2$. Solid line shows the run with deceleration phase and dashed line that without deceleration.
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FIG. 8. (a) \([L_O]/h_0\) vs \([L_T]/h_0\) diagram. (b) \(\Gamma_c\) vs \([R_{OT}]\) diagram. Runs are with the deceleration phase are plotted. Open circle is \(Re_0 = 300\) the pairing is observed (AD1-AD4, AD7-AD9 and AD+). Cross is the same as open circle but no pairing (AD5-AD6, AD10-AD16 and AD-). Triangle is \(Re_0 = 500\) or 800 (ADR1, ADR2 and ADR+). Diamond is the long forcing cases (ADR1+, ADL+ and ADRI+). Dotted lines are \(R_{OT} = 0.79\) (Dillon 1982) and \(R_{OT} = 0.66\) (Crawford 1986). Solid lines in (a) are \(R_{OT} = 0.5\) and \(R_{OT} = 1.0\). Dashed line in (b) is \(R_{BT}^2 = 0.1\) by Garrett (2001).
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