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These notes correspond to the course "OC599: Oregon Coast Math Camp" (formerly "Math on the Beach"), taught by the College of Oceanic and Atmospheric Sciences at Oregon State University. This is a 1.5-week residential course, held at the Hatfield Marine Science Center in Newport, Oregon. The course covers the basic methods of applied mathematics needed by beginning graduate students in the natural sciences. The lectures are designed to accompany a sequence of science presentations and field trips through which the class explores the spectacular and fascinating environment of the Oregon coast.

The intent of this course is that students in math-intensive disciplines should get a useful review of the most important methods as well as seeing them developed in the context of applications in the natural sciences. Students in less math-intensive programs will gain familiarity with the advanced mathematical concepts that form the “language” of natural science, while not necessarily becoming expert in their use.

Acknowledgements: These notes owe much to the help of Bob Miller and Martin Hoecker-Martinez. Thanks also to the class of 2009, the first to try this course, whose questions, comments and alertness to error revealed many opportunities for improvement. Since that class the notes have been greatly expanded and clarified. Further suggestions are welcome.

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1 Common threads

As you follow this course, notice the following recurring themes.

Techniques

- Differentiating scalar and vector functions
- Differential equations: solving, interpreting the solution in terms of underlying dynamics
- Eigenvalues/vectors/functions: these apply to both matrices and differential operators

Principles These arise as methods for solving differential equations, but are really general philosophies for thinking about how nature works.

- Separation: Divide a phenomenon into separate aspects and try to understand each individually.
- Superposition: A simple process, replicated on a range of sizes, frequencies and orientations, can model the complexity of natural systems.

Objectives Not all, but a large fraction of the mathematical methods we use to understand nature involve the analysis of equilibria:

- Equilibria are the states a system may be in where there is no change in time. Such states can be good models for systems that evolve slowly in the context of human perceptions.
- Perturbation analysis: determining the stability (or lack thereof) of an equilibrium state. Stable equilibria form the foundations for the wide variety of waves and oscillatory phenomena found in nature. Unstable equilibria are the source of turbulence, mixing and various natural catastrophes.
2 Day 1: Spring tides and sneaker waves

Ocean waves and tides pervade every aspect of life near the ocean, and we will therefore begin with a quick discussion of the mathematics of these phenomena. Waves and tides are examples of a broader class of phenomena called oscillations. A quantity is said to oscillate when it changes in a repeating pattern, or cycle.

Most oscillations can be described by trigonometric functions (e.g. the well-known "sine wave"), so we will begin this section with a quick review of trigonometry. We will then use the properties of trig functions to understand some essential properties of waves and tides.

2.1 First, some terminology

A function is a process that converts one number into another, hopefully for some useful purpose. A simple example would be \( C(F) = \frac{5}{9}(F - 32) \), which takes the Fahrenheit temperature \( F \) and converts it to the Celsius temperature \( C \). The input variable, \( F \) in this case, is called the argument to the function. Another term for a quantity that goes into a function is the independent variable. That is because the function is supposed to work for any value of the input variable\(^1\), so the input variable can vary independently of any other constraint. For example, \( C(F) = \frac{5}{9}(F - 32) \) gives the Celsius equivalent of any Fahrenheit temperature you may care to choose, so \( F \) is considered independent. \( C \), on the other hand, is the dependent variable because its value depends on the value inserted for \( F \).

Many everyday activities can be thought of as functions, e.g. computing a tip at a restaurant (the tip depends on the total bill), doubling the quantities in a recipe (the quantity used depends on the original quantity), or estimating how long it will take to drive somewhere (the time depends on the distance). Think of a few others.

We often find it useful to represent a function as a line or curve on a graph, as in figure 1.

\(^1\)or at least over some range of values
Figure 1: A function of the independent variable $x$. A special case of a function is a straight line with slope $m$ and $y$-intercept $b$.

Figure 2: Trigonometric sine and cosine functions of the argument $\phi$. 
Besides making it easier to find values of the function, this helps us to understand how the function “functions” by engaging our visual intuition.

### 2.2 Trigonometric functions

The basic functions of trigonometry, the sine and the cosine, are oscillatory functions as shown in figure 2. They are defined as sides of a right triangle inscribed within a circle of unit radius (figure 3). The properties of these functions are described by the identities that connect them. Some of these are easy to deduce from the unit circle diagram. For example

\[
\sin^2 \phi + \cos^2 \phi = 1, \quad \text{or} \quad \sin \left( \frac{\pi}{2} - \phi \right) = \cos \phi.
\]

Many other properties of trigonometric functions will be deduced later in this course (mostly by you). For now, the only one we need is the addition formula for the sin function:

\[
\sin(a) + \sin(b) = 2 \sin \left( \frac{a+b}{2} \right) \cos \left( \frac{a-b}{2} \right).
\] (2.1)

Pick any values you like for \( a \) and \( b \) and try this on your calculator.
\section*{2.3 Oscillations as trig functions}

A fundamental property of any oscillation is its \textit{period}, $T$, the shortest time interval between identical cycles. The frequency, $f = 1/T$, is the number of cycles per unit time, e.g. per second. Another fundamental property is the amplitude, half the difference between the maximum and minimum value (for the case of ocean waves, this is just the height of the wave crests above mean sea level).

Most oscillations that occur in nature are quite well described by functions like

$$h(t) = h_0 \sin(2\pi ft), \quad \text{or} \quad h(t) = h_0 \sin\left(2\pi \frac{t}{T}\right),$$

\begin{equation}
(2.2)
\end{equation}

where the constant $h_0$ is the amplitude. As shown in figure 2, the sin function is a cyclic oscillation, like a wave. Starting at $t = 0$, it rises to a maximum, falls and becomes negative for the second half of its cycle, then returns to zero when $t = T$.

\section*{2.4 Beats}

The “sneaker waves” that visitors to the Oregon coast are warned about, the spectacular spring tides (which have nothing to do with the season), and a musician’s ability to hear when a string is exactly in tune are all examples of the phenomenon of \textit{beats} (figure 4). When two oscillations with slightly different frequencies $f_1$ and $f_2$ occur together, what we perceive is a single oscillation with frequency equal to the average $(f_1 + f_2)/2$, modulated in amplitude by a slower frequency equal to the difference $|f_1 - f_2|$. Here, we’ll see how these results follow from the addition rule (2.1) for the sine function.

For the present application, we identify $a$ and $b$ as $2\pi f_1 t$ and $2\pi f_2 t$, so that (2.1) can be written as:

$$\sin(2\pi f_1 t) + \sin(2\pi f_2 t) = 2 \sin\left(2\pi \frac{f_1 + f_2}{2} t\right) \cos\left(2\pi \frac{f_1 - f_2}{2} t\right).$$

\begin{equation}
(2.3)
\end{equation}

The right hand side describes an oscillation with the average frequency $(f_1 + f_2)/2$, modulated

\footnote{Though the modeling of oscillations by trig functions is familiar to most of us, it may not be exactly clear why this works. What does the ratio of sides of a right triangle have to do with waves? This will also become clear in the pages to follow.}
Figure 4: Two oscillations summed such that every seventh wave is especially large. The dashed curve shows the envelope function \( \cos(x/14) \).

by the slower frequency \( (f_1 - f_2)/2 \). The slower oscillation is called the *envelope*, and is shown by the dashed curve in figure 4.

A subtlety to notice is that the oscillation has maximum amplitude when the envelope \( \cos(2\pi \frac{f_1 - f_2}{2} t) \) is equal to *either* +1 or -1. Beats therefore occur with double the frequency of the envelope: \( 2 \times |(f_1 - f_2)/2| = |f_1 - f_2| \), or just the absolute difference between the two frequencies. For example, in figure 4, the oscillation \( \sin(x) \) is modulated by the envelope \( \cos(x/14) \). The envelope has frequency 1/14, so the beat frequency is 1/7.

**Example 1**: To tune a string, one begins by playing the string together with another oscillation having the correct pitch. If the string is slightly out of tune, the ear perceives beats. The tension in the string is then adjusted in whichever direction makes the beats become slower. As the string approaches the correct frequency, the beat frequency approaches zero. You know the string is in tune when the beats are no longer heard.
3  Day 2: Univariate calculus

Calculus is the science of change. Does a thing stay the same, or does it change? Does it change at a constant rate, or is the rate itself increasing or decreasing? If you know how something changes, what does that tell you about the thing itself?

Calculus is a set of operations performed on functions. In this section, we apply the methods of calculus to functions of a single variable, like \( C(F) = \frac{5}{9}(F - 32) \) or the examples in figure 1. Later we will deal with functions of multiple variables.

3.1  The derivative

Slope is a measure of steepness. In figure 1, the slope of the straight line is constant, but the slope of the function \( f(x) \) varies with \( x \); the curve is steeper in some places than others. At any given \( x \), the slope of \( f(x) \) is defined as the slope of the tangent line at that point (figure 5). This is also the geometrical definition of the derivative of the function \( y = f(x) \) with respect to \( x \). The derivative is written as \( f'(x) \), or as \( dy/dx \) (compare with the definition of the slope \( m \) in figure 1).

Alternatively we can define the derivative as the limiting slope of a line connecting two points as the two points converge:

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},
\]

as illustrated in figure 6. This definition allows us to compute the derivative algebraically.

**Example 2:** If \( f(x) = x^2 \), then:

\[
\frac{f(x+h) - f(x)}{h} = \frac{(x^2 + 2hx + h^2) - x^2}{h} = 2x + h
\]

so that

\[
f'(x) = \lim_{h \to 0} 2x + h = 2x.
\]
Figure 5: The tangent line intersects $f(x)$ at a single point.

Figure 6: As $h \to 0$, the line connecting $f(x)$ and $f(x + h)$ rotates to become the tangent line at $x$ whose slope is the derivative $f'(x)$. 
An alternative notation for the derivative is

\[ f'(x) \equiv \frac{df}{dx}. \]

While it takes more writing than the "prime" symbol, this notation reminds us of what the derivative is: a ratio of small changes in \( x \) and \( f \). The example above is a special case of the power rule for differentiation:

\[ \frac{d}{dx} x^\alpha = \alpha x^{\alpha - 1}, \quad (3.2) \]

where \( \alpha \) is any constant.

Also useful are the sum rule and the product rule for two functions \( f(x) \) and \( g(x) \):

\[ (f + g)' = f' + g' \quad (3.3) \]
\[ (fg)' = f'g + fg'. \quad (3.4) \]

The product rule gives a simple result when one of the functions (\( g \), say) is just a constant:

\[ (af)' = af'. \quad (3.5) \]

Combining (3.3) and (3.5), we see that differentiation is a linear operation, for applying it to a linear combination of two functions yield the same linear combination of the derivatives:

\[ (af + bg)' = af' + bg'. \quad (3.6) \]

Note that (3.3) and (3.5) are special cases of (3.6).

### 3.2 Composite functions and the chain rule

In practical examples, it is often true that the argument of a function is itself a function of some other variable, e.g. \( y = f(x); \; x = g(t) \). The resulting form \( y = f(g(t)) \) is called a composite function.

**Example 3**: Suppose you drive north from the equator. Your latitude (call it \( \phi \)) is a function of the distance you have traveled (call it \( y \)). Moreover, because you are moving, the distance \( y \) is a function of time, \( t \), so that \( \phi = f(y(t)) \).
Can you calculate your rate of change of latitude with respect to time? Latitude is proportional to distance from the equator, $\phi = y/y_0$. If latitude is in degrees then $y_0$ is about 110km. Suppose you drive at a constant velocity $V = 100$ km/hr so that $y = Vt$. Then your speed of 100 km/hr is multiplied by 1 degree/110 km to get $100/110 = 0.91$ degrees per hour.

In mathematical language:

$$\frac{d\phi}{dt} = \frac{d\phi}{dy} \frac{dy}{dt} = \frac{\text{1 degree}}{\text{110 km/hr}} \frac{100 \text{ km}}{\text{hr}} = 0.91 \frac{\text{degrees}}{\text{hr}}.$$ 

(Note: In this calculation we have carefully retained the units and ensured that they cancel properly. The student should be in the habit of doing this, as it would otherwise be easy to get the ratio of units upside-down.)

The foregoing is an example of the chain rule: If $y = f(x)$; $x = g(t)$, then

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = f'(x)g'(t). \quad (3.7)$$

Notice that example 3 is nothing but a conversion of units from kilometers to degrees. Unit conversions are an excellent example of the chain rule; in fact many if not most applications of the chain rule can be thought of as conversions between different ways of measuring the rate of change of some quantity.

### 3.3 Higher-order derivatives

The derivative of a function is itself a function, so it makes sense that it too would have a derivative. The second derivative is just that - the derivative of the derivative. If

$$f'(x) = \frac{df}{dx},$$

then

$$f''(x) = \frac{d^2f}{dx^2} = \frac{d}{dx} \frac{df}{dx}.$$ 

**Example 4**: Let $f(x) = x^2$. Then

$$f'(x) = 2x$$
Figure 7: \( R \) is the radius of a circle such that both the circle and its 1st and 2nd derivatives match \( f \) at the point \( x \). \( 1/|K| \), where \( K \) is the curvature of \( f \).

and

\[ f''(x) = 2. \]

Example: A common example of a second derivative is the acceleration of an object. If \( x(t) \) is the object’s position, then the velocity is \( v(t) = x'(t) \) and \( a(t) = v'(t) = x''(t) \).

Higher-order derivatives, e.g. \( f'''(x) \), \( f''''(x) \) are defined analogously. When primes become profuse, we use a superscript in parentheses, e.g. \( f''''(x) = f^{(4)}(x) \).

The second derivative tells us about the curvature of a function: if \( f'' > 0 \) the function bends upward (or the slope \( f' \) is increasing), whereas if \( f'' < 0 \) the function bends downward (or \( f' \) is decreasing). More specifically, the curvature \( K \) is defined by

\[ K = \frac{f''}{(1 + f'^2)^{3/2}}. \]

\( |K|^{-1} = R \) is the radius of curvature which is the radius of a circle tangent to \( f(x) \) as shown in figure 7.
A zero of the derivative indicates an extremum of the function, which can in most cases be identified as either a maximum or a minimum by the sign of the second derivative as shown in figure 8. An inflection point, where \( f'' = 0 \), is an extremum of \( f' \) and can be identified as a maximum or a minimum (of \( f' \)) by the sign of \( f''' \).

### 3.4 Taylor series

The Taylor series expansion of a function \( f(x) \) about the point \( x_0 \) can be written as follows:

\[
\tilde{f}(x) = f(x_0) + f'(x_0)h + \frac{1}{2!} f''(x_0)h^2 + \frac{1}{3!} f'''(x_0)h^3 + \cdots + \frac{1}{n!} f^{(n)}(x_0)h^n + \cdots, \quad (3.8)
\]
where \( h = x - x_0 \). The factorial function \( n! \) that appears in (3.8) is defined by

\[
\begin{align*}
0! &= 1 \\
1! &= 1 \\
2! &= 2 \times 1 = 1 \\
3! &= 3 \times 2 \times 1 = 6 \\
&\vdots \\
n! &= n(n-1)(n-2)\cdots 1.
\end{align*}
\]

(3.9)

If the Taylor series (3.8) converges at any point \( x \), then \( \tilde{f}(x) = f(x) \) (except in a few weird cases that we won’t worry about). Taylor series expansions are most commonly used with only a few terms retained to approximate a function in a small neighborhood of \( x_0 \).

**Example 5:** Construct the Taylor series expansion of \( f(x) = (1 - x)^{-1} \) about \( x = 0 \). The first few derivatives are

\[
\begin{align*}
f'(x) &= (1 - x)^{-2}; \\
f''(x) &= 2 \cdot (1 - x)^{-3}; \\
f'''(x) &= 3 \cdot 2 \cdot (1 - x)^{-4}.
\end{align*}
\]

Evidently \( f^{(n)}(x) = n! \cdot (1 - x)^{-(n+1)} \), hence \( f^{(n)}(0) = n! \). The Taylor series (figure 9) is just

\[
(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^n.
\]

This gives us a useful approximation to remember: \((1 - x)^{-1} \approx 1 + x \) for \( x \ll 1 \).

Note that the Taylor series for \((1 + x)^{-1}\) can be obtained from this result simply by changing \( x \) to \( -x \), i.e. by changing the signs of the odd-powered terms:

\[
(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n.
\]

**3.5 Differentiating data**

In both field observations and numerical modeling, we often find ourselves needing to estimate the derivative of a function from a finite list of values \( f_n = f(x_n); n = 1, 2, \cdots, N \). Most

---

3The Taylor series for the special case \( x_0 = 0 \) is often called the McLaurin series, but we won’t bother with that distinction here.

---
commonly the $x_n$ are evenly spaced, i.e. $x_n = nh$, where $h$ is a constant increment of $x$. The most obvious method is called the forward difference (FD):

$$f_n' \approx \frac{f_{n+1} - f_n}{h}.$$  

This is nothing but the definition of the derivative (6) with the limit $h \to 0$ ignored. A close relative is the backward difference (BD):

$$f_n' \approx \frac{f_n - f_{n-1}}{h}.$$  

For finite $h$, of course, these approximations will involve some error. Taylor series help us to understand these errors better. Rearranging (3.8), the forward difference can be written as

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{1}{2!} f''(x)h + \frac{1}{3!} f^{(3)}(x)h^2 + \frac{1}{4!} f^{(4)}(x)h^3 + \cdots \quad (3.10)$$

If $h$ is sufficiently small, then the main source of error is the second term on the right-hand side. To see this, note that $h^2 \ll h$, and $h^3$ and the higher powers that come later are even smaller. The presence of $f''$ in the main error term suggests that this error has to do with curvature. To see the connection more clearly, refer to figure 10, where the slope of the red line corresponds to the forward difference. Because the function curves upward ($f'' > 0$), the approximated slope is too steep. Equivalently, the main error term in (3.10) is positive.
Figure 10: Slopes approximated using the (red) forward difference and (blue) backward difference formulas. The black line is the actual slope.

The backward difference can also be written using the Taylor series form:

\[
\frac{f(x) - f(x-h)}{h} = f'(x) - \frac{1}{2!}f''(x)h + \frac{1}{3!}f^{(3)}(x)h^2 - \frac{1}{4!}f^{(4)}(x)h^3 + \cdots \quad (3.11)
\]

The sign of the main error term is now reversed. This corresponds to the fact that, in the example of figure 10, a slope given by the backward difference approximation is too shallow.

Since the forward and backward difference approximations err in different directions, one might guess that a more accurate approximation might be obtained by taking the average of the two, and the Taylor series shows that this is true. Adding (3.10) and (3.11) and dividing by 2, we obtain:

\[
\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{1}{3!}f^{(3)}(x)h^2 + \cdots.
\]

The error term due to curvature has canceled out. The main error term is now the relatively small term proportional to \( h^2 \). This approximation is called a centered difference (CD). For practical use, we write it as

\[
f'_n \approx \frac{f_{n+1} - f_{n-1}}{2h} + O(h^2).
\]

The final term reminds us that the dominant error is proportional to the square of the increment \( h \). In this case, we say that the approximation is accurate to second order in the increment.
While we’re at it, let’s try subtracting (3.10) and (3.11):

\[
\frac{f(x+h) - 2f(x) + f(x-h)}{h} = \frac{2}{2!} f''(x)h + \frac{2}{4!} f^{(4)}(x)h^3 \cdots ,
\]

or

\[
f''_n \approx \frac{f_{n+1} - 2f_n + f_{n-1}}{h^2} + O(h^2).
\]

This is a centered difference approximation for the second derivative. Once again, the dominant error term is proportional to \(h^2\), so the approximation is accurate to second order. Higher order approximations can be derived with just a little more algebra, but (3.12) and (3.14) are used most often in practice.

### 3.6 Day 3: Antiderivatives and integrals

Suppose we have a function \(g(x)\) and its derivative is \(g'(x) = f(x)\). In that case, \(g(x)\) is called an antiderivative of \(f(x)\). Because the derivative of a constant is zero, any other function gotten by adding a constant to \(g\) is also an antiderivative. Another (less informative) term for the antiderivative is the indefinite integral. We generally write the antiderivative as

\[
g(x) = \int f(x)dx + C,
\]

where \(C\) is a constant.

Suppose now that we want to compute the area under the curve \(y = f(x)\) from \(x = x_1\) to \(x = x_2\). If \(f(x)\) were a constant, this would be easy; the area would just be \(f \times (x_2 - x_1)\). Instead, we break \(x\) up into a series of intervals of width \(h\) along which \(f\) can be approximated by constant, as shown in figure 11a.

\[
\text{Area} \approx \sum_n f_nh.
\]

The area becomes exact in the limit as \(h\) goes to zero (or as the number of intervals goes to infinity). This is called the definite integral of \(f(x)\) from \(x = x_1\) to \(x = x_2\):

\[
\int_{x_1}^{x_2} f(x)dx = \lim_{h \to 0} \sum_{n=1}^{N} f_nh.
\]

If an antiderivative \(g\) is available, the definite integral is the difference between values of \(g\) at the endpoints:

\[
\int_{x_1}^{x_2} f(x)dx = g(x_2) - g(x_1) = g|_{x_1}^{x_2}.
\]
Figure 11: Approximation of the area under $f$ by (a) a series of rectangles and (b) a series of trapezoids.

Standard calculus texts describe many ways of determining the antiderivative of a given function. The strategy in all of these is to manipulate the given function into a form recognizable as the derivative of something, so that the “something” is the antiderivative we need. We will not go into these methods here, but the student is encouraged to review them as necessary.

3.7 Integrating data

As with the derivative, we often need to approximate the integral from a series of values of $f$. The method defined in (3.16), and illustrated in figure 11a, is the equivalent of the forward difference approximation to the derivative. It is easily implemented but not very accurate. The analog of the more accurate centered difference is called the trapezoidal rule. As the name suggests, it requires that the area under the curve be broken up into a series of trapezoids, as shown in figure 11b. In that case the approximation becomes

$$\text{Area} \approx \sum_{n=1}^{N} \frac{1}{2} (f_n + f_{n+1})h = \frac{h}{2} f_1 + h \frac{f_{N-1}}{2} + \sum_{n=2}^{N-1} f_n + \frac{h}{2} f_N. \quad (3.17)$$

This is almost as easy to implement as (3.16). As the second form shows, it’s just (3.15) with the first and last terms halved.
The solution of the differential equation requires the starting point, \( g(x_0) \), and the slope \( g' = f(x) \) at every \( x \).

**3.8 Differential equations**

An *algebraic* equation involves functions, and its solutions are numbers, e.g. \( x^2 - 4 = 0 \). A *differential* equation involves derivatives, and its solutions are functions. The simplest differential equation is one we’ve already seen: \( g' = f \), where \( f \) is given. The solution for \( g \) is the antiderivative of \( f \).

Because the antiderivative is only defined up to an additive constant, the complete solution requires an additional piece of information, the value of \( g \) at some point \( x_0 \): \( g(x_0) = g_0 \). The solution is then

\[
g(x) = g_0 + \int_{x_0}^{x} f(\chi) d\chi.
\]

In this example the independent variable \( x \) is the upper limit of the integral. The variable of integration is a *dummy* variable. A dummy variable can be called anything; here I’ve called it \( \chi \). Check for yourself that \( g(x) \) satisfies the equation \( g' = f \) and the additional condition \( g(x_0) = g_0 \).

The additional piece of information needed to specify the solution, \( g_0 = g(x_0) \), is either called a *boundary condition* or an *initial condition*, depending whether the independent variable represents space or time. Here we have called the independent variable \( x \), which suggests the former.
Many useful theorems and solution methods work only for a specific class of differential equations, so we start by defining some of these classes.

- **Linear** equations contain the solution function and its derivatives only in linear form, e.g. added together or multiplied by given functions, not multiplied together, raised to any power, or subjected to any other abuse:

  \[
  f''' - 3f = 0 \quad \text{linear}
  \]
  \[
  f''' - 3f^2 = 0 \quad \text{nonlinear}
  \]
  \[
  f'f'' + f = 0 \quad \text{nonlinear}
  \]
  \[
  f' + 2f = 1/f \quad \text{mondo nonlinear!}
  \]

- A **homogeneous** equation has \( f(x) = 0 \) as a solution:

  \[
  f''' - 3f = 0 \quad \text{homogeneous}
  \]
  \[
  f''' - 3f^2 = 0 \quad \text{homogeneous}
  \]
  \[
  f'f'' + f = 1 \quad \text{nonhomogeneous}
  \]

Linear, homogeneous equations obey the principle of **superposition**, which is a lot more useful than you might think. It says that if \( f(x) \) and \( g(x) \) are solutions, then any linear combination \( af(x) + bg(x) \), where \( a \) and \( b \) are constants, is also a solution.

- The **order** of an equation is the order of its highest derivative, for example:

  \[
  f' - g = 0 \quad \text{1st order}
  \]
  \[
  f''' - 3f^2 = 0 \quad \text{3rd order}
  \]
  \[
  f'f'' + f = 0 \quad \text{2nd order}
  \]
  \[
  f' + 2f = 1/f \quad \text{1st order}
  \]

For a linear equation, the number of linearly independent solutions (i.e. not linear combinations of each other) is equal to the order.

- Finally, an **ordinary** differential equation (or ODE) contains only total derivatives, like the ones we’ve been using so far. This is to distinguish it from a **partial** differential equation (PDE), which contains partial derivatives. We’ll get to those later.
Figure 13: The solution of (3.18) must have both function value and derivative equal to 1 at $x = 0$. As we move to the right, the function increases, so the slope must also increase. The further we move to the right, the steeper the slope becomes. Similarly, as we move to the left from $x = 0$, the slope must decrease, eventually approaching zero.

3.9 Exponential functions

Here we study the exponential function as an example of the way that differential equations are analyzed. We will see that an ODE can pack a huge amount of information into a small space, and that the methods of calculus allow us to extract that information.

We begin by pretending that we know nothing of exponential functions but are given a simple, first-order ODE and boundary condition:

$$f'(x) = f(x); \quad f(0) = 1.$$  \hspace{1cm} (3.18)

The equation doesn’t have a solution that’s any simple power of $x$, the only kind of function we’ve differentiated so far (try it). What can we deduce about the solution? We can learn a lot by making a sketch, as illustrated in figure 13. Alternatively, we could write a computer program to approximate the solution of (3.18) by replacing the derivative with a finite-difference approximation, which would give us a plot much like figure 13.

Other properties of the solution can be deduced using the methods of calculus that we’ve covered so far. To start with, let’s give the solution a name: $E(x)$. Now note that we can
differentiate both sides of the ODE in (3.18), which shows that $E'' = E'$. We can repeat the process to find that $E''' = E''$, etc., and therefore all derivatives are equal to $E$. At $x = 0$ the situation is even simpler: all derivatives are equal to 1. This property makes it easy to write $E$ in the form of a Taylor series about $x = 0$ (cf. equation 3.8):

$$E(x) = E(0) + E'(0)x + \frac{1}{2!}E''(0)x^2 + \frac{1}{3!}E^{(3)}(0)x^3 + \cdots + \frac{1}{n!}E^{(n)}(0)x^n + \cdots$$

(3.19)

It's not much harder to write $E(x+y)$ as a Taylor series about $x$:

$$E(x+y) = E(x) + E'(x)y + \frac{1}{2!}E''(x)y^2 + \frac{1}{3!}E^{(3)}(x)y^3 + \cdots + \frac{1}{n!}E^{(n)}(x)y^n + \cdots$$

$$= E(x) \left\{ 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots + \frac{1}{n!}y^n + \cdots \right\}$$

$$= E(x)E(y).$$

Thus we have a rule that connects addition and multiplication: $E(x+y) = E(x)E(y)$.

How about exponentiation? Is there anything we can learn about an object like $E^2$? To answer this, simply set $y = x$ in our addition rule: $E(x+x) = E(x)E(x)$, or $E(x)^2 = E(2x)$. By setting $y = 2x$, we get $E(x)^3 = E(3x)$ (try it!) and so on to higher powers. Thus we have our rule for exponentiation: $E(x)^\alpha = E(\alpha x)$. A special case of this is $E(-x) = 1/E(x)$.

We also have the differentiation rule $dE(\alpha x)/dx = \alpha E$, which is easily obtained using the chain rule.

You should recognize by now that $E$ is just the familiar exponential function $e^x$, where $e = 2.71728 \cdots$. Knowing only that this function is the solution of (3.18), we have deduced both the shape of the function and the following well-known properties:

$$e^{x+y} = e^xe^y,$$

(3.20)

$$\left(e^x\right)^\alpha = e^{\alpha x}, \text{ with special case } e^{-x} = 1/e^x,$$

(3.21)

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots, \text{ with special case } e^1 = 1 + 1/2 + \cdots \approx 2.71828$$

(3.22)

$$\frac{d}{dx}e^{\alpha x} = \alpha e^{\alpha x}, \text{ and therefore}$$

$$\int e^{\alpha x}dx = \alpha^{-1}e^{\alpha x} + C.$$  

(3.23)
Read these examples to see how exponential functions can arise in practice:

**Example 6:** Newton’s law of cooling states that an object warmer than its surroundings radiates heat at a rate proportional to the temperature difference, i.e.

\[
\frac{dT}{dt} = -k(T - T_0)
\]  

(3.25)

where \( T \) is the temperature of the object, \( T_0 \) is the (constant) temperature of the surroundings \( t \) is time and \( k \) is a diffusion constant. Because the derivative of \( T_0 \) is zero, we can rewrite the equation as

\[
\frac{d(T - T_0)}{dt} = -k(T - T_0).
\]  

(3.26)

Comparing with (3.23), we see that the solution is

\[ T - T_0 = Ce^{-kt}, \]

Because equation (3.26) is linear and homogeneous with respect to \( T - T_0 \) the solution can always be multiplied by a constant, hence the constant \( C \). Suppose the initial condition is \( T(0) = T_1 \). This gives us the value of \( C \) and allows us to complete the solution:

\[ T - T_0 = (T_1 - T_0)e^{-kt}. \]

The difference in temperature between object and surroundings decreases exponentially, eventually approaching zero.

**Example 7:** A sand grain of mass \( m \), having been washed into the ocean by a river, begins to sink. Its vertical velocity \( w(t) \) is given by

\[
\frac{dw}{dt} = -\frac{m}{m_0}g - kw,
\]  

(3.27)

where \( m_0 \) is the mass of water displaced by the sand grain, \( g = 9.8m/s^2 \) is the gravitational acceleration and \( k \) is a friction constant. The first term on the right-hand side describes the effect of **buoyancy**: if \( m > m_0 \) (the particle is denser than water), the particle sinks; otherwise, it floats. We can simplify the math by dividing through by \( m \) and rearranging:

\[
\frac{dw}{dt} + \frac{k}{m}w = -g',
\]  

(3.28)
where $g' = g(1 - m_0/m)$ is called the \textit{reduced gravity}. We will solve (3.28) by the use of an \textit{integrating factor}. Using the product rule of differentiation and property (3.23) of the exponential function, we note that

$$\frac{d}{dt}(we^{\frac{k}{m}t}) = \frac{dw}{dt}e^{\frac{k}{m}t} + \frac{k}{m}we^{\frac{k}{m}t}.$$ 

Thus we multiply (3.28) through by $e^{\frac{k}{m}t}$ and rearrange to get

$$\frac{d}{dt}(we^{\frac{k}{m}t}) = -g'e^{\frac{k}{m}t}.$$ 

Taking the antiderivative of both sides, using property (3.24) of the exponential function, gives

$$we^{\frac{k}{m}t} = -g'm\frac{k}{m}e^{\frac{k}{m}t} + C$$

If we assume that $w(0) = 0$, i.e. time is measured from the moment the sand grain started to sink, then we can fix the value of the constant $C$ and obtain

$$w = -g'm\frac{k}{m} \left(1 - e^{-\frac{k}{m}t}\right).$$

Note that, as $t \to \infty$, $e^{-\frac{k}{m}t} \to 0$, so the particle approaches its \textit{terminal velocity} $w = -g'm/k$.

In the exercises for this section, you will see how exponential functions arise in compound interest and carbon dating. You will also use the same analytical methods to study another simple ODE and derive some other very useful functions. Do this before proceeding to the next subsection.

\section*{3.10 Day 4: Complex exponential, trigonometric and hyperbolic functions}

Complex numbers are basically a book-keeping trick that makes some kinds of calculations immensely easier. A complex number $z$ has the form $z = z_r + iz_i$, where the numbers $z_r$ and $z_i$ are just ordinary real numbers and are called the real and imaginary parts of $z$. The symbol $i$ is the fundamental imaginary number, defined by $i = \sqrt{-1}$. The \textit{complex conjugate} of a
complex number is gotten by reversing the sign of the imaginary part or, if you prefer, of \(i\):
\[ z^* = z_r - iz_i. \]
The definitions of \(z\) and \(z^*\) can be solved for \(z_r\) and \(z_i\) to get:
\[
z_r = \frac{z + z^*}{2}; \quad z_i = \frac{z - z^*}{2i}.
\] (3.29)

Here we will compute the real and imaginary parts of the complex exponential function \(e^{ix}\).
We evaluate the Taylor series (3.19) for the argument \(ix\):
\[
E(ix) = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 \ldots
\]
\[
= 1 + ix + \frac{1}{2!}i^2x^2 + \frac{1}{3!}i^3x^3 + \frac{1}{4!}i^4x^4 + \frac{1}{5!}i^5x^5 + \ldots
\] (3.30)

From the definition \(i = \sqrt{-1}\), we can work out all of the powers of \(i\): \(i^2 = -1, i^3 = i^2 \times i = -i, i^4 = 1, i^5 = i\) and so on. Substituting these, we have
\[
E(ix) = 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 - i\frac{1}{5!}x^5 + \ldots
\]
\[
= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots\right) + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots\right)
\] (3.31)

The real and imaginary parts of \(E(ix)\) are in parentheses. From the previous problem set (section 3.9, exercise 4g,h) we recognize these as the Taylor series expansions for \(C(x)\) and \(S(x)\), the cosine and sine functions. We thus have a relationship between \(E, C\) and \(S\): \(E(ix) = C(x) + iS(x)\). Using the more familiar terminology:
\[
e^{ix} = \cos x + i\sin x.
\] (3.32)

The complex conjugate of \(e^{ix}\) is gotten, as usual, by reversing the sign of \(i\), which has the effect of reversing the sign of the imaginary part: \(e^{-ix} = \cos x - i\sin x\). Like any other complex quantity, \(e^{ix}\) can be solved for its real and imaginary parts in terms of itself and its complex conjugate (compare with (3.29)):
\[
\cos x = \frac{e^{ix} + e^{-ix}}{2}; \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.
\]

The hyperbolic sine and cosine functions are combinations of positive and negative exponentials:
\[
\sinh(x) = \frac{e^x - e^{-x}}{2}; \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.
\]
There are also hyperbolic tangent, secant and other functions, defined analogously with trigonometric functions, e.g.:

\[
\tanh x = \frac{\sinh x}{\cosh x} \quad \text{sech} x = \frac{1}{\cosh x}
\]

(see examples in figure 14). The hyperbolic sine and cosine are easily seen to obey this relation:

\[
\cosh^2 x - \sinh^2 x = 1,
\]

which leads to other useful relations like \(\text{sech}^2 x = 1 - \tanh^2 x\).

### 3.11 Newton’s 2nd law for small oscillations

In a recent exercise we used the differential equation (9.4) to derive the properties of two functions \(S\) and \(C\) that we easily recognize as the sine and cosine functions of trigonometry. We’ll now show that this equation represents a broad class of oscillatory phenomena. Newton’s second law of motion, the famous \(F = ma\), can be written

\[
\frac{d^2 y}{dt^2} = F
\]  

(3.33)
where $m$ is the mass of an object, $y$ is its position and $F$ is the net force acting on it. The second derivative is the acceleration, or $d/dt$ of the velocity $dy/dt$. It is often the case that the force is a function of position, $F = F(y)$, and can therefore be expanded as a Taylor series, about $y = 0$, say: $F(y) = F(0) + F'(0)y + \cdots$. Suppose that $y = 0$ is a position of equilibrium, at which no force acts. In that case $F(0) = 0$, hence $F(y) \approx F'(0)y$ for $y$ sufficiently close to 0. Then we can rewrite Newton’s law as

$$m\frac{d^2y}{dt^2} = F'(0)y,$$

(3.34)

Finally, suppose $F$ has the character of a restoring force, i.e. if the object moves away from the equilibrium position $x = 0$, the force acts to push it back (e.g. the action of gravity on a pendulum). For these cases, $F'(0) < 0$. We can now write Newton’s second law as

$$\frac{d^2y}{dt^2} + \omega^2y = 0, \quad \text{where } \omega^2 = \frac{-F'(0)}{m}.$$

(3.35)

The equation is equivalent to (3.36). With the change of variables $\tau = \omega t$, $d^2y/dt^2$ can be rewritten as $\omega^2d^2y/d\tau^2$, and (3.35) is equivalent to (9.4). Its solutions are therefore the sine and cosine functions of $\tau$, or $\sin \omega t$ and $\cos \omega t$. The constant $\omega$ is the angular frequency, and is equal to $2\pi f$, or $2\pi/T$.

The solution $y(t)$ needn’t represent spatial position; it could just as easily be pressure or electromagnetic field strength, for example. The trig functions are therefore a model for any of the multitude of oscillations that arise in nature: light waves, sound waves, brain waves and, of course, ocean waves.

### 3.12 Differential eigenvalue problems

Consider the following equation and boundary conditions:

$$f''(x) + \lambda^2 f(x) = 0;$$

(3.36)

$$f(0) = 0; \quad f(\pi) = 0.$$

(3.37)

This is called a two-point boundary value problem, because the boundary conditions are specified at two different values of $x$. The equation is familiar; its general solution is

$$f = A \sin(\lambda x) + B \cos(\lambda x)$$

(3.38)
where $A$ and $B$ are arbitrary constants.

Substituting (3.38) into the first of the boundary conditions (3.37), we see that $B$ must equal zero, so $f = A \sin(\lambda x)$. Next, we substitute this into the second boundary condition, and get:

$$A \sin(\pi \lambda) = 0.$$  

In general, there is no solution to this except the trivial solution $A = 0$, or $f(x) = 0$ for all $x$. Nontrivial solutions exist only if $\sin(\pi \lambda) = 0$, which is true if and only if that the constant $\lambda$ is an integer.

The values $0, \pm 1, \pm 2, \cdots$ are the **eigenvalues** of $\lambda$: the special values for which (3.36,3.37) has a solution. For each of these eigenvalues, there is a solution $f = A \sin(\lambda x)$; this is called the **eigenfunction** corresponding to the particular eigenvalue.

This nomenclature is reminiscent of the matrix eigenvalue problem, and in fact (3.36) has that form if you think of $f$ as a vector and $\partial^2 / \partial x^2$ as a matrix. The analogy is a useful one. We’ve already seen that a function is often represented as a list of values, either in a numerical code or as the result of a series of measurements. Now if you take the second derivative using a finite difference approximation such as (3.14), this is equivalent to multiplying a matrix onto $f$. 

**Figure 15:** Eigenmodes for the system (3.36, 3.37). Eigenvalues are $\lambda = 1, 2, 3, 4$ as indicated in the legend.
An alternative term for eigenfunctions of a differential eigenvalue problem is *eigenmodes* or just *modes*. These provide a useful model for trapped waves. Examples include waves in an enclosed basin (e.g. the resonantly amplified tides in the Bay of Fundy, Nova Scotia) or internal waves bounded between the ocean bottom and the surface. Higher order modes (corresponding to higher eigenvalues) exhibit more rapid spatial oscillations, as illustrated in figure 15.
4 Multivariate calculus

4.1 Multivariate functions and partial derivatives

A good example of a function of several variables is sea surface temperature (SST). Perhaps most obviously, it varies in the meridional (north-south) direction, being warmer near the equator and colder near the poles. SST also varies in the zonal (east-west) direction. In the Pacific, for example, the SST at the equator is colder in the east and warmer in the west, and this difference works with the trade winds to drive El Nino. So if we call the eastward direction $x$ and the northward direction $y$, we can write SST as $T = f(x, y)$. We’re not done yet, though. SST also changes over time, e.g. due to the rising and setting of the sun and the passage of storms, so we should write it as $T = f(x, y, t)$.

The choice of independent variables is not carved in stone. For example, if we were working near a coastline that was not conveniently oriented north-south (like the Oregon coast), we might use a rotated coordinate system $x', y'$ so that one of the coordinates would be parallel to the coast. In that case we’d write $T = f(x', y', t)$. The functional dependence on the coordinates would be different (hopefully simpler), but the value of the temperature at a given place and time would be the same in either representation.

In this section we’ll cover derivatives of multivariate functions, called partial derivatives, and the partial differential equations (PDE’s) that allow us to compute and understand many natural phenomena. We’ll look mainly at examples with only two independent variables, e.g. a spatial coordinate $x$ and time $t$. Later in the course we’ll look at more general cases.

A partial derivative is just a derivative taken with respect to one independent variable while holding the other independent variable(s) constant:

$$\frac{\partial}{\partial x} f(x, t) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, t) - f(x, t)}{\Delta x}; \quad \frac{\partial}{\partial t} f(x, t) = \lim_{\Delta t \to 0} \frac{f(x, t + \Delta t) - f(x, t)}{\Delta t}. \quad (4.1)$$
If both $x$ and $t$ change by a small increment, the corresponding increment in $f$ is given by

$$\Delta f = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial t} \Delta t. \quad (4.2)$$

An important special case is the case where $x$ is itself a function of $t$, e.g. if we’re making measurements from a moving vessel $x = Vt$. We divide (4.2) by $\Delta t$ and take the limit $\Delta t \to 0$ to obtain:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial t}. \quad (4.3)$$

The left-hand side is written as a total derivative because all sources of variability in time are accounted for. The first term on the right-hand side describes the change in $f$ due to the change in $x$ with respect to time. In our example of the moving ship, this is the change due to the ship motion, and could also be written as $V \partial f / \partial t$ (cf. exercise 3 in section 2). The derivative $dx/dt$ is a total derivative because, once again, all sources of variation in time are accounted for (trivially in this case because $x$ depends only on $t$). The second term on the right-hand side describes the intrinsic variability of $f$ with respect to time. In the case of sea surface temperature, this might represent warming by the sun, which happens independently of the motion of our ship.

A useful diagnostic in the natural sciences is the isocontour. An isocontour is a surface in space along which the value of a function is constant, e.g. $f(x, y) = \text{const}$. The slope of an isocontour can be deduced from the fact that the increment $\Delta f$ is zero:

$$\Delta f = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y = 0.$$  

Rearranging, we have

$$\frac{dy}{dx} = - \frac{\partial f / \partial x}{\partial f / \partial y}.$$  

**Example 8**: Figure 16a shows temperature as a function of latitude and depth in the Pacific ocean. The warmest temperatures are at the surface near the equator; the coldest are at the bottom toward the Southern Ocean. Figure 16b shows salinity in the same way, indicating three water masses that flow into the Pacific as detailed in the caption. The slopes of the isotherms are:

$$\left( \frac{dz}{d\phi} \right)_T = - \frac{\partial T / \partial \phi}{\partial T / \partial z},$$

and likewise for salinity. The subscript on the left-hand side indicates that the derivative is taken along a curve where $T$ is a constant.
Figure 16: Measurements from a north-south section through the western Pacific Ocean. (a) Isotherms (curves of constant temperature) show warm surface water near the equator. (b) Isohalines (constant salinity) indicate three prominent water masses flowing into the Pacific: The Antarctic Intermediate Water, the Pacific Sub-polar Intermediate Water (largely from from the Bering Sea) and the Antarctic Bottom Water.

4.2 Day 5: Partial differential equations

Partial differential equations, or PDEs, are simply differential equations that involve partial derivatives. PDEs are classified according to order, linearity and homogeneity just like ODEs. In the following example we write partial derivatives using the more compact subscript notation.

\[ f_x = \frac{\partial f}{\partial x}; \quad f_{tt} = \frac{\partial^2 f}{\partial t^2}; \quad f_{xxt} = \frac{\partial^3 f}{\partial x \partial t^2}, \ldots \]

\[
\begin{align*}
  f_t + f_{xx} - 3f &= 0 \quad \text{linear} \\
  f_x f_t - 3f^2 &= 0 \quad \text{nonlinear} \\
  f_x + f_t^2 - 3f &= 0 \quad \text{homogeneous} \\
  f_x + f_t^2 - 3f &= 1 \quad \text{nonhomogeneous}
\end{align*}
\]

Just as with ODEs, PDEs that are **linear** and **homogeneous** obey the principle of **superposition**: if \( f(x,t) \) and \( g(x,t) \) are solutions, then any linear combination \( af(x,t) + bg(x,t) \), where \( a \) and \( b \) are constants, is also a solution.
The order is determined by the highest derivative with respect to any variable:

\[
\begin{align*}
f_x + f_t - g &= 0 \quad \text{1st order} \\
f_{xx} + f_t - 3f^2 &= 0 \quad \text{3rd order}
\end{align*}
\]

4.3 Solution of PDEs by partial integration

Suppose \( f_x(x, y) = x \). What can we learn about \( f \)? We can compute an antiderivative with respect to \( x \) as usual: \( f(x, y) = \frac{1}{2}x^2 + \cdots \), but instead of a constant of integration we must add an unknown function \( c(y) \) to get the general solution. Check by differentiating that \( f(x, y) = \frac{1}{2}x^2 + c(y) \) solves our original equation. Suppose we also have a boundary condition, e.g. \( f(0, y) = 1 + y \). We can substitute this to identify the unknown function \( c(y) = 1 + y \), giving the complete solution.

In some cases the right hand side of the equation is not a function of the variable in the derivative, e.g. \( f_x(x, y) = y \). The solution is again obtained by integrating with respect to \( x \), treating \( y \) as a constant: \( f(x, y) = xy + c(y) \). In fact, the solution to \( f_x(x, y) = g(y) \), where \( g \) is any function, is just \( f(x, y) = g(y)x + c(y) \).

As a final example, let \( f_x(x, y) = 0 \). The general solution to this is \( f = c(y) \). Thus \( f_x = 0 \) is another way of saying that \( f \) does not depend on \( x \).

4.4 Solution of PDEs by separation of variables

Often a PDE can be solved, or at least greatly simplified, by assuming the solution is the product of functions of the individual independent variables, e.g. \( f(x, t) = X(x)T(t) \). In this case, with a little luck, you can reduce the problem to a pair of ODEs.

The method requires answering the following riddle: "Under what conditions can two functions of different variables be equal?" For example, suppose \( f \) is a function of \( x \) and \( g \) is a function of \( y \). Is it possible that \( f(x) = g(y) \) for all \( x \) and \( y \)? Think about this for a minute before you read on.
A good answer would be "No." For suppose you changed the value of $x$ while leaving $y$ the same. This would change the value of $f$ while leaving $g$ the same, so how could $f$ and $g$ be equal? Actually, depending on the functions involved, there might be a particular way to vary $x$ and $y$ that would make $f = g$. For example, if $f(x) = x$ and $g(y) = y^2$, then $f$ and $g$ would be equal along the curve $x = y^2$. But for all $x$ and $y$? No way.

Well, there is one way. A very simple type of function is the constant function $f(x) = C$. If $C$ can be regarded as a function of $x$, then it can just as easily be regarded as a function of any other variable, including $y$. So if $f(x) = C$ and $g(y) = C$, then $f(x) = g(y)$ for all $x$ and $y$. Like most answers to riddles, this one may seem a bit glib, but it’s exactly what we need for the separation of variables technique.

Consider the following PDE:

$$f_t + f_{xx} = 0.$$  

Let’s try a solution of the form $f(x, t) = X(x)T(t)$:

$$X(x)T_t(t) + X_{xx}(x)T(t) = 0.$$  

Now divide through by $XT$ and rearrange a little:

$$\frac{T_t(t)}{T(t)} = -\frac{X_{xx}(x)}{X(x)}.$$  \hspace{1cm} (4.4)  

The left-hand side is purely a function of $t$ while the right-hand side is purely a function of $x$. From our previous discussion we know that this can only be true if both functions are equal to a constant:

$$\frac{T_t}{T} = -\frac{X_{xx}}{X} = C.$$  \hspace{1cm} (4.5)  

We don’t know what the constant is; we’ll have to figure that out based on whatever initial/boundary conditions we are given. But we have succeeded in separating our PDE into a pair of ODES:

$$T_t = CT; \quad X_{xx} = -CX.$$  

The ODEs are both easy to solve, and the result looks like this

$$f(x, t) = X(x)T(t) = [A \sin(\sqrt{C}x) + B \cos(\sqrt{C}x)]e^{Ct},$$
where $A$ and $B$ are constants. If we can find values of $A$, $B$ and $C$ such that our solution satisfies the initial and boundary conditions, then we are done.

Here’s a more realistic example. Consider a rod extending from $x = 0$ to $x = L$. The temperature of the rod is $T(x,t)$. The ends are maintained at a temperature we’ll call $T = 0$. At some initial time $t = 0$, the temperature in the rod is $T_0(x)$. We’ll specify the form of this function later. The problem is to find $T(x,t)$.

Thermal diffusion is governed by the diffusion equation

$$ T_t = \kappa T_{xx}, \quad (4.6) $$

where the constant $\kappa$ is called the diffusivity. The boundary conditions are

$$ T(0,t) = 0; \quad T(L,t) = 0, \quad (4.7) $$

and the initial condition is

$$ T(x,0) = T_0(x), \quad (4.8) $$

where $T_0(x)$ is some given function.

Try a solution of the form $T(x,t) = X(x)\Theta(t)$. Substituting and rearranging as before, we get:

$$ \frac{\Theta_t}{\Theta} = \kappa \frac{X_{xx}}{X} = C, \quad (4.9) $$

so that

$$ \Theta_t = C\Theta; \quad X_{xx} = \frac{C}{\kappa}X. $$

At this stage we choose the sign of $C$. Assume $C$ is positive, in which case we write it as $C = a^2$ where $a$ is a real constant. The solutions are then

$$ \Theta = e^{a^2 t}; \quad X = A e^{\frac{a}{\sqrt{\kappa}}x} + B e^{-\frac{a}{\sqrt{\kappa}}x}. $$

This solution will not work. The boundary conditions require that $X(0) = 0$ and $X(L) = 0$. We can choose the constants $A$ and $B$ to satisfy one of these conditions, but not both. (Try it!) Moreover, the temporal part of the solution grows exponentially, so the temperature of the bar will become infinitely hot (or cold)! We have better luck if we assume that $C$ is negative, i.e. $C = -a^2$ where $a$ is again some real constant. The solutions are now

$$ \Theta = e^{-a^2 t}; \quad X = A \sin \frac{a}{\sqrt{\kappa}}x + B \cos \frac{a}{\sqrt{\kappa}}x. $$
The boundary condition $X(0) = 0$ requires that $B = 0$. The condition $X(L) = 0$ is equivalent to $\sin \frac{aL}{\sqrt{\kappa}} = 0$. This constrains the possible values of $a$:

$$\frac{aL}{\sqrt{\kappa}} = n\pi; \quad \text{where } n = 0, \pm 1, \pm 2, \cdots.$$  

At this stage our solution is

$$T(x,t) = X \Theta = A \sin \left( \frac{a_n}{\sqrt{\kappa}} x \right) e^{-a_n^2 t}; \quad \text{where } a_n = \frac{\pi \sqrt{\kappa}}{L} \{0, \pm 1, \pm 2, \cdots\},$$

which satisfies both the PDE (4.6) and the boundary conditions (4.7).

Now let’s think about initial conditions. Certain choices are obviously convenient. For example, suppose

$$T_0(x) = \bar{T} \sin \left( \frac{m\pi}{L} x \right)$$

where $m$ is some integer. Then the solution just requires that we choose $n = m$ in (4.10). A few of points to notice:

- As $t \to \infty$, the solution goes to zero. This means the temperature in the bar eventually equalizes to the (fixed) temperature of the endpoints.
- The rate at which the temperature decays to zero is $a_n^2$, which is proportional to the diffusivity $\kappa$. The larger the diffusivity, the faster temperature fluctuations diffuse.
- The rate $a_n^2$ at which the temperature decays is also proportional to $n^2$, or to the inverse square of the length scale of the initial fluctuation (see figure 15 and the accompanying discussion).

The final point above will become clearer if we consider another initial condition:

$$T_0(x) = \bar{T}_1 \sin \left( \frac{\pi}{L} x \right) + \bar{T}_3 \sin \left( \frac{3\pi}{L} x \right), \quad (4.12)$$

where $\bar{T}_1$ and $\bar{T}_3$ are constants. Thanks to the principle of superposition, the solution is just

$$T(x,t) = \bar{T}_1 \sin \left( \frac{\pi}{L} x \right) e^{-\frac{\pi^2}{L^2} t} + \bar{T}_3 \sin \left( \frac{3\pi}{L} x \right) e^{-\frac{9\pi^2}{L^2} t}.$$  \hspace{1cm} (4.13)

The second fluctuation (with $m = 3$) has one-third the wavelength and decays nine times as fast.
Of course, in real life we can’t just choose initial conditions for our convenience as we’ve done here. BUT, as we’ll see in the next subsection, any function $T_0(x)$ that obeys the boundary conditions can be represented as a sum of sine functions like (4.12). Therefore the diffusion equation can be solved for any initial condition. This is thanks to the principle of superposition, which I told you would be important.

Separation of variables does not work for all PDEs. The key to its use in (4.5) and (4.9) was that we were able to manipulate the PDE into a statement that a function of one variable equals a function of the other variable. There are some other tricks for doing this, such as

- choosing variables differently, and
- making approximations valid in particular parameter regimes.

Sometimes, though, no matter how you try, a PDE will not separate. Those PDEs are called (not surprisingly) nonseparable.

### 4.5 Fourier series

We’ll now consider the problem from the previous subsection for the general initial condition $T(x,0) = T_0(x)$. Taking superposition into account, we construct a general solution by replacing (4.10) with

$$T(x,t) = \sum_{n=1}^{N} A_n \sin \left( \frac{n\pi x}{L} \right) e^{-\frac{n^2\pi^2}{L^2} t},$$

where the $A_n$ are constants to be determined. This is the generalization of (4.13), a sum over all values of $n$ with a couple of exceptions. First, we don’t use $n = 0$ because it just gives $\sin(0)=0$. Second, we don’t use negative values of $n$ because the sine functions are just the negatives of their partners with $n > 0$, so they don’t add anything. This is another way of saying they’re not linearly independent.

Now at $t = 0$, (4.14) becomes

$$T(x,0) = \sum_{n=1}^{N} A_n \sin \left( \frac{n\pi x}{L} \right)$$
which has to equal $T_0(x)$:
\[ \sum_{n=1}^{N} A_n \sin \left( \frac{n\pi}{L} \right) = T_0(x). \] (4.15)

The initial conditions (4.11) and (4.12) are special cases of this general form. Fourier’s Theorem assures us that ANY initial condition that satisfies the boundary conditions can be represented in this way. As with the Taylor series, the more terms you keep, the more precise the representation. The remaining challenge is to compute the coefficients $A_n$. A general formula for the purpose is derived below.

**Calculating coefficients for the Fourier series**

We begin by multiplying (4.15) by $\sin \left( \frac{m\pi}{L} x \right)$ and integrating over the length of the bar:
\[ \sum_{n=1}^{N} A_n \int_{0}^{L} \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{n\pi}{L} x \right) \, dx = \int_{0}^{L} \sin \left( \frac{m\pi}{L} x \right) T_0(x) \, dx \] (4.16)

noting that both the summation on the left hand side and the constants $A_n$ are independent of $x$, so the integral passes through them. To do the integral on the left hand side, we use a trig identity similar to the one you’ve derived:
\[
\sin a \sin b = \frac{1}{2} \left[ \cos(a - b) - \cos(a + b) \right].
\]

Thus:
\[ \int_{0}^{L} \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{n\pi}{L} x \right) \, dx = \int_{0}^{L} \left[ \cos \left( \frac{(m-n)\pi}{L} x \right) - \cos \left( \frac{(m+n)\pi}{L} x \right) \right] \, dx = \frac{L}{2\pi} \int_{0}^{\pi} \left[ \cos(m-n)y - \cos(m+n)y \right] \, dy,
\]

after the change of variables $y = \frac{\pi}{L} x$. Now, provided $m \neq n$, the first term integrates easily to give:
\[ \int_{0}^{\pi} \cos(m-n)y \, dy = \frac{\sin(m-n)y}{m-n} \bigg|_{0}^{\pi} = 0. \]

On the other hand, if $m = n$, the first term is
\[ \int_{0}^{\pi} \cos(0) \, dy = \pi. \]

The second term is simpler because $m$ and $n$ are both positive, so $m + n \neq 0$. The term integrates to zero in the same way as the first term. Finally we have
\[ \int_{0}^{L} \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{n\pi}{L} x \right) \, dx = \frac{L}{2} \left\{ \begin{array}{ll} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{array} \right\} \]
Substituting this into (4.16), we find a huge simplification: all terms in the summation over $n$ are zero except the single term where $n = m$! Finally we have

$$A_m = \frac{2}{\pi} \int_0^L \sin \left( \frac{m \pi x}{L} \right) T_0(x) dx. \quad (4.17)$$

The expansion (4.16), with the $A$’s given by (4.17), is the Fourier series for the function $T_0(x)$. It converges and is equal to $T_0$ as long as $T_0$ is finite and goes to zero at $x = 0$ and $x = L$. Also, with this choice of the $A$’s, (4.14) is the complete solution for the original problem (4.6,4.7,4.8) for any initial condition $T_0(x)$. 

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5 Day 6: Linear algebra

5.1 Vectors and matrices

We review some facts about vectors and matrices. The following should be familiar to you from elementary algebra courses. Read carefully, then do the exercises.

- A scalar $a$ is a single number.
- A vector is, in the simplest definition, just a list of numbers.

\[
\vvec = \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_N
\end{pmatrix}, \quad \text{Row vector } \vvec = (v_1, v_2, \cdots v_N).
\]

$v_i$ is the $i$’th component of $\vvec$, for $i = 1, 2, 3, \cdots, N$.

- Vector sum: $\vvec = \uvec + \vvec$, or $w_i = u_i + v_i$, for $i = 1, 2, 3, \cdots, N$.

- Scalar multiplication: $\uvec = a\vvec$, or $u_i = av_i$.

- Dot (or scalar) product: $\uvec \cdot \vvec = \sum_{i=1}^{N} u_i v_i$.

- Einstein summation notation: $\uvec \cdot \vvec = u_i v_i$; summation over the repeated index is implied. The repeated index is called a dummy index.

- Magnitude (or length, or absolute value) of a vector: $|\vvec| = \sqrt{\vvec \cdot \vvec} = \sqrt{v_i v_i} = \sqrt{v_i^2}$.

- The dot product (or inner product, or scalar product) of $\uvec$ and $\vvec$ can be written in terms of the magnitudes of the two vectors and the angle between them, $\theta$: $\uvec \cdot \vvec = |\uvec||\vvec|\cos \theta$.

- Orthogonal vectors: $\cos \theta = 0 \Rightarrow \uvec \cdot \vvec = 0$. The vectors are at right angles, i.e. $\theta = \pm \pi/2$. 
Figure 17: Component of $\vec{u}$ in the direction of $\vec{v}$.

- Unit vector $\hat{e} = \vec{v}/|\vec{v}|$. $|\hat{e}| = 1$. $\hat{e}$ is parallel to $\vec{v}$.
- Component of $\vec{u}$ in the direction of $\vec{v} = \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|}$ (see figure 17).
- A matrix is a 2D array of numbers, e.g.

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

The first subscript indicates the row; the second indicates the column.
- Matrix transpose: $A^T_{ij} = A_{ji}$.
- Symmetric matrix: $A^T = A$ or $A_{ji} = A_{ij}$.
- Antisymmetric matrix: $A^T = -A$ or $A_{ji} = -A_{ij}$.
- Diagonal matrix: $A_{ji} = 0$ unless $i = j$.
- Matrix addition: $C_{ij} = A_{ij} + B_{ij}$.
- Matrix multiplication: $C = AB$. For 2x2 case:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

or $C_{ij} = A_{ik}B_{kj}$ (sum on middle index). In general, matrix multiplication is not commutative, i.e. $AB \neq BA$.
- In the equation $C_{ij} = A_{ik}B_{kj}$, $k$=dummy index, $i,j$=free indices. For a matrix equation to be consistent, free indices on the left and right-hand sides must correspond, i.e. the matrices on the two sides must be the same shape.
- Identity matrix:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{e.g.} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
• Matrix *inverse*: If $AB = \delta$ and $BA = \delta$ then $B = A^{-1}$ and $A = B^{-1}$.

• *Orthogonal* matrix: $A^{-1} = A^T$.

• The *determinant* of a matrix is gotten by multiplying diagonals, then adding the products of diagonals oriented downward to the right (red in figure 18) and subtracting the products of diagonals oriented upward to the right (blue in figure 18), for example:

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb.$

• The determinant can also be evaluated via expansion of cofactors along a row or column. In this example we expand along the top row:

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

• *Singular* matrix: $|A| = 0$. A singular matrix has no inverse.

• Matrix-vector multiplication: $\vec{u} = A\vec{v}$ or $u_i = A_{ij}v_j$. Note the sum is on the second, or right index of $A$.

• Matrix-vector multiplication can be done from either side. The case described above is *right multiplication*. *Left* multiplication works oppositely: $\vec{u} = \vec{v}A$ means $u_j = v_iA_{ij}$. Note that the sum is now on the *first*, or left, index of $A$.

• $\vec{u} = A\vec{v}$, or $A\vec{v} = \vec{u}$, represents a set of linear equations that can (usually) be solved for $\vec{v}$. Provided $|A| \neq 0$, then $\vec{v} = A^{-1}\vec{u}$. 

*Figure 18*: Groups of elements to be multiplied when computing the determinant.
• A homogeneous set of equations has the form \( A\vec{v} = 0 \), i.e. it has \( \vec{v} = 0 \) as a solution (like homogeneous differential equations). In this case, nonzero solutions for \( \vec{v} \) exist only if \( |A| = 0 \).

• The cross product (or vector product) of two vectors gives a third vector: \( \vec{u} \times \vec{v} = \vec{w} \). The magnitude \( |\vec{w}| \) is given by \( |\vec{u}||\vec{v}||\sin \theta \), where \( \theta \) is the angle separating the two vectors. The direction is perpendicular to both \( \vec{u} \) and \( \vec{v} \), in the sense specified by the right-hand rule\(^4\).

• The cross product of 3-element vectors \( \{u_1,u_2,u_3\} \) and \( \{v_1,v_2,v_3\} \) can be evaluated as

\[
\vec{u} \times \vec{v} = \{u_2v_3 - v_2u_3, -u_1v_3 + v_1u_3, u_1v_2 - v_1u_2\}.
\]

• In case you have trouble remembering the foregoing expression, here is a mnemonic that takes advantage of the fact that a cross product is related to the determinant of a matrix:

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix} = \hat{i}(u_2v_3 - v_2u_3) - \hat{j}(u_1v_3 - v_1u_3) + \hat{k}(u_1v_2 - v_1u_2)
\]

In this expression, \( \hat{i} \), \( \hat{j} \) and \( \hat{k} \) are unit vectors that serve to identify the the three components of the cross product.

• The following are useful relationships between dot and cross products of four vectors \( \vec{u}, \vec{v}, \vec{w} \) and \( \vec{x} \):

\[
\begin{align*}
\vec{u} \cdot (\vec{v} \times \vec{w}) &= (\vec{u} \times \vec{v}) \cdot \vec{w} & (5.1) \\
\vec{u} \times (\vec{v} \times \vec{w}) &= (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} & (5.2) \\
(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{x}) &= (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{x}) - (\vec{u} \cdot \vec{x})(\vec{v} \cdot \vec{w}) & (5.3) \\
(\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v}) &= (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2 & (5.4)
\end{align*}
\]

**Example 9:** We close this subsection with an only somewhat fanciful example of an application of a set of linear equations to be solved in matrix form. We would like to know the direction in which sand moves along a beach. We surmise

\(^4\)If right-hand fingers curl from \( \vec{u} \) to \( \vec{v} \), thumb points to \( \vec{w} \).
Figure 19: A beach has sand made primarily of two minerals, and the percentages of the two have been carefully measured. We assume that the beach is fed mainly by sediments of two nearby rivers. Each river’s sediment has mineral content characteristic of the rocks in its own drainage basin, and we have measured the proportions of our two minerals in each. The mathematical challenge is to determine the percentage of each river’s characteristic sediment that makes up the beach sand, and hence the average direction of sand motion.
that the sand comes from two rivers, one to the north and one to the south, and would therefore like to know which source is dominant. For this, we analyze the mineralogical content of the beach sand and of the sediments characteristic of each river, then combine that information to form a linear, algebraic system to tell us the desired proportions.

Writing the percentage of each mineral in the beach as a linear combination of the contributions from each river, we have

\[
\begin{pmatrix}
0.6 \\
0.4
\end{pmatrix}
= C \begin{pmatrix}
0.9 \\
0.1
\end{pmatrix}
+ R \begin{pmatrix}
0.4 \\
0.6
\end{pmatrix}
\]

which we can easily rearrange as a linear matrix equation for \( C \) and \( R \):

\[
\begin{pmatrix}
0.9 & 0.4 \\
0.1 & 0.6
\end{pmatrix}
\begin{pmatrix}
C \\
R
\end{pmatrix}
= \begin{pmatrix}
0.6 \\
0.4
\end{pmatrix}
\]

This is easily solved using the standard prescription for inversion of a 2x2 matrix. We interchange the components on the main diagonal, change the signs of the off-diagonal components, and divided the result by the determinant:

\[
\begin{pmatrix}
C \\
R
\end{pmatrix}
= \frac{1}{0.9 \times 0.6 - 0.1 \times 0.4}
\begin{pmatrix}
0.6 & -0.4 \\
-0.1 & 0.9
\end{pmatrix}
\begin{pmatrix}
0.6 \\
0.4
\end{pmatrix}
= \begin{pmatrix}
0.4 \\
0.6
\end{pmatrix}
\]

We find that 40% of the beach sand comes from the Northern source, 60% from the Southern, i.e. the sand flux is bidirectional but mainly from the South.\(^5\)

5.2 Cartesian position vectors

One application of vectors is to denote points in three-dimensional, physical space. Any point, measured from some given origin, is denoted by a vector which has both a length and a direction. There are many ways to define such a vector; the most common is the Cartesian coordinate system.

A unit vector is a vector of unit length. Here, we denote unit vectors by a hat rather than an arrow, e.g. \( \hat{i} \) is a vector such that \( \hat{i} \cdot \hat{i} = 1 \). A Cartesian coordinate system\(^6\) is based on three

\(^5\)Real beach sand is a mixture of many minerals, typically more than the number of nearby rivers. The problem therefore has more equations than unknowns and hence cannot be solved exactly. A least-squares approach can be used to identify the solution that comes closest to satisfying all of the equations.

\(^6\)Invented by the philosopher Rene Descartes.
unit vectors emanating from some fixed point in space (called the origin of the system) at right angles to one another. We usually specify that the orientation be right-handed, meaning that, if you curl the fingers of your right hand to point from the direction of the first unit vector to the direction of the second, your thumb will point in the direction of the third.

Any point in space can be specified by a position vector which is a linear combination of the three unit vectors: \( \vec{p} = x\hat{i} + y\hat{j} + z\hat{k} \). The coefficients in this linear combination are the coordinate values \( \{x, y, z\} \). In the geosciences, a common choice is to have \( \hat{i} \) point to the east and \( \hat{j} \) to the north. Test your understanding: for the system to be right-handed, which direction must \( \hat{k} \) point?

Another common nomenclature for Cartesian unit vectors is \( \hat{e}(n) \), where \( n = 1, 2, 3 \). Other useful coordinate systems include cylindrical (radial-azimuthal-vertical), which is useful for studying vortices such as tornados, and spherical (longitude-latitude-altitude), useful for large-scale motions on a sphere such as a planet.
5.3 Matrices as transformations

A $3 \times 3$ matrix $A$ can be thought of as a transformation that acts on position vectors, written as $\vec{v}' = A\vec{v}$ or $\vec{v} \rightarrow A\vec{v}$. Such a transformation generally involves a change in direction (a rotation) and a change in length (a magnification).

Example 10:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \vec{v} \rightarrow \vec{v}.$$  

This is the null transformation. Note that $|A| = 1$.

Example 11:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \vec{v} \rightarrow 2\vec{v}.$$  

$A$ magnifies any vector by the factor 2. Note that $|A| = 8$. Also note that this transform is reversible: if I give you the “new” vector and the transformation used to get it, you can deduce the “old” vector.

Example 12:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \rightarrow \begin{bmatrix} v_2 \\ v_1 \\ v_3 \end{bmatrix}.$$  

$A$ interchanges the first and second components of any vector. The transformation is reversible. What is the determinant?

Example 13:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \rightarrow \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}.$$  


A sets the third component to zero. The transformation is not reversible! Information about \( v_3 \) is irrevocably lost. Note \(|A| = 0\). This is generally true for irreversible transformations. Multiplication by the inverse \( A^{-1} \) reverses the transformation, and \( A^{-1} \) does not exist when \(|A| = 0\).

### 5.4 Eigenvalues and eigenvectors

Recall example 10 in the previous subsection:

\[
A = \begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{bmatrix} \Rightarrow \vec{v}' = 2\vec{v}.
\]

The direction of the vector is unchanged. This is always true for certain vectors, called eigenvectors:

\[
\vec{v}' = A\vec{v} = \lambda\vec{v}.
\]

The scalar \( \lambda \) that indicates change in magnitude is called an eigenvalue. If \( A \) is an \( N \times N \) matrix, then there will be \( N \) eigenvalue-eigenvector pairs:

\[
A\vec{v}^{(n)} = \lambda^{(n)}\vec{v}^{(n)} \quad \text{(no sum on } n\text{)}.
\]

To find \( \lambda^{(n)} \) and \( \vec{v}^{(n)} \):

\[
A\vec{v}^{(n)} = \lambda^{(n)}\delta\vec{v}^{(n)} \Rightarrow (A - \lambda^{(n)}\delta)\vec{v}^{(n)} = 0.
\]

This is a homogeneous set of equations, so it can only have a solution if the determinant is zero:

\[
|A - \lambda^{(n)}\delta| = 0. \quad (5.5)
\]

If \( A \) is an \( N \times N \) square matrix, then (5.5) can be written as an \( N \)'th order polynomial for \( \lambda \). It has \( N \) solutions, though not necessarily distinct ones.

**Example 14:** Example

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
sets the third component of any vector to zero. To find the eigenvalues:

\[ |A - \lambda \delta| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(1 - \lambda^2) = 0. \]

This has three solutions, two of which are equal: \( \lambda = 0, 1 \) and 1.

Now for the eigenvectors. First we’ll find the eigenvectors corresponding to \( \lambda = 1 \). These are vectors such that both direction and length are unchanged after multiplication by \( A \). From the original homogeneous equation \( A \vec{v} = \lambda \vec{v} \) we have

\[ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = 1 \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \]

or \( v_1 = v_1, \ v_2 = v_2 \) and \( 0 = v_3 \). This tells us that \( v_1 \) and \( v_2 \) can be anything, but \( v_3 \) has to be zero. In other words the matrix that changes \( v_3 \) to zero has no effect on direction if \( v_3 \) is already zero.

Now how about the case \( \lambda = 0 \)?

\[ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = 0 \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \]

or \( v_1 = 0, \ v_2 = 0 \) and \( 0 = 0 \). So \( v_1 \) and \( v_2 \) must be zero, but \( v_3 \) is arbitrary. It doesn’t matter what \( v_3 \) is because it just gets squashed to zero.
6  Day 7: Linear systems of ODEs

This section concerns systems of first order, linear, homogeneous, ordinary differential equations. As this might seem like a rather restrictive class of equations to concentrate on, we begin by showing how higher order equations can be reduced to first order. Next, we’ll see that the first step toward understanding a nonlinear system is to render it linear. The methods described in this section also extend to homogeneous and partial differential systems, but we will not pursue those extensions here.

6.1 Reduction of order

Consider the second order equation \( f_{xx} = -f \). To reduce this equation to first order, define \( g \) such that \( f_x = g \). We now have two equations:

\[
\begin{align*}
f_x &= g \\
g_x &= -f,
\end{align*}
\]

or

\[
\frac{d}{dx} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}
\]

(6.2)

The 2nd order equation has been replaced by a pair of 1st order equations. In general, any equation of order greater than one can be reduced to a set of 1st order equations in this way, i.e. by defining derivatives as separate functions.

6.2 Nonlinear systems

Many phenomena are described by coupled sets of differential equations. An example is the famous “hares and foxes” problem, which is a useful approximation for many predator-prey systems (e.g. whales and plankton). Let the population of hares and foxes be given by \( H(t) \)
and $F(t)$. Hares, if left alone, would multiply at a rate $r$, whereas foxes alone would starve to death at a rate $s$. When the two interact, hares get eaten while foxes’ survival prospects are enhanced:

$$\frac{dH}{dt} = rH - aFH$$
$$\frac{dF}{dt} = -sF + bFH. \quad (6.3)$$

The constants $r$, $a$, $s$ and $b$ are assumed to be positive (or zero).

This is a relatively hard system to solve because it is, like most systems in the real world, nonlinear. The way we address such a system is to explore its equilibrium states, the states at which $dH/dt = dF/dt = 0$. We consider the stability of each equilibrium state. If an equilibrium is stable, you can expect to see the system in that state often. In contrast, an unstable equilibrium is almost never observed, but the character of the instability tells you something about the kinds of behavior you can expect to see.

The system (6.3) has two equilibria:

$$F = H = 0 \quad (6.4)$$

and

$$F = r/a; \quad H = s/b. \quad (6.5)$$

Choose one of these equilibrium states and label it $F_0, H_0$. The states close to the equilibrium can be represented as $F = F_0 + F'$, $H = H_0 + H'$. The primes indicate small perturbations (not derivatives!). The product $FH$ is then written as

$$(F_0 + F')(H_0 + H') = F_0H_0 + F_0H' + H_0F' + F'H' \approx F_0H_0 + F_0H' + H_0F'. \quad (6.6)$$

The expansion is truncated just as we have done before with Taylor series, with the assumption that, if $F'$ and $H'$ are small, then the product $F'H'$ is even smaller.

We substitute these results into the original system and get:

$$\frac{dH'}{dt} = (r - aF_0)H' - aH_0F'$$
$$\frac{dF'}{dt} = -(s - bH_0)F' + bF_0H'. \quad (6.7)$$

Although it looks a little more complicated than (6.3), this system is linear and homogeneous, and can therefore be solved in a straightforward way to determine whether the perturbations
increase or decrease in time. This tells about the stability or instability of the equilibrium \( F_0, H_0 \). The easy case is the state \( F_0 = H_0 = 0 \). Substitute this into (6.7) and you will see immediately that \( H \) grows exponentially while \( F \) decreases exponentially. In other words, if you have just a few hares and a few foxes, the hares will multiply but the foxes will starve.

The second equilibrium state is not so easy to solve; you’ll do it for homework after you’ve learned how. In the rest of this section we’ll see how to solve linear systems of ODEs like (6.7) and thus explore the nature of equilibria.

### 6.3 Linear, homogeneous systems

The most general linear, homogeneous system can be written as

\[
\frac{d\vec{v}}{dt} = A\vec{v}
\]

(6.8)

where \( \vec{v} \) is a vector and \( A \) is a matrix. For example, the perturbation equation (6.7) for the hares and foxes problem can be written as

\[
\frac{d}{dt} \begin{pmatrix} H' \\ F' \end{pmatrix} = \begin{pmatrix} r - aF_0 & -aH_0 \\ bF_0 & -s + bH_0 \end{pmatrix} \begin{pmatrix} H' \\ F' \end{pmatrix}.
\]

Let us seek a solution of (6.8) having the form \( \vec{v} = \vec{v}_0 e^{\sigma t} \), where the vector \( \vec{v}_0 \) is independent of time\(^7\). In that case (6.8) becomes

\[
A\vec{v}_0 = \sigma \vec{v}_0,
\]

(6.9)

which simply says that \( \vec{v}_0 \) must be an eigenvector of \( A \) and \( \sigma \) is the corresponding eigenvalue.

The eigenvalue \( \sigma \) represents an exponential growth rate. In general, \( \sigma \) is complex: \( \sigma = \sigma_r + i\sigma_i \), and stability is governed by the signs of the real and imaginary parts. Figure 21 shows five evolutionary patterns that the solution can follow. The solution is unstable, i.e. it grows exponentially in time, if \( \sigma_r > 0 \). Conversely, if \( \sigma_r < 0 \), the solution decays and the system is stable. If \( \sigma_i \neq 0 \), the growth or decay is modulated by an oscillation of frequency \( \sigma_i \). If \( \sigma \) is purely imaginary, the system oscillates without growing or decaying; this is called a neutrally stable oscillation.

\(^7\)This is another example of the concept of “separaton”. The solution is a time-dependent vector, but it’s directional aspect and its time-dependent aspect are expressed as two factors to be solved for separately.
Figure 21: Examples of stable, unstable and neutrally stable exponential solutions of a single solution with complex growth rate $\sigma$. 
Figure 22: Equilibria of a two-equation system with solution vector $\vec{v} = \{v_1, v_2\}$.

Cases a, b and c have two real eigenvalues. In cases d, e and f eigenvalues form a complex conjugate pair, with equal real parts and opposite imaginary parts.

The general solution is

$$\vec{v}(t) = a_1 \vec{v}_0^{(1)} e^{\sigma_1 t} + a_2 \vec{v}_0^{(2)} e^{\sigma_2 t} + \cdots = \sum_{n=1}^{N} a_n \vec{v}_0^{(n)} e^{\sigma_n t}, \quad (6.10)$$

where the $\vec{v}_0^{(n)}$ are the eigenvectors with corresponding eigenvalues $\sigma_n$. The $a_n$ are constants whose values are determined by the initial conditions$^8$.

(6.10) looks complicated, but the most important thing to recognize is that, if all eigenvalues have negative real parts, the solution decays to zero, but if even one eigenvalue has a positive real part, the solution will eventually grow. In the latter case, the solution will eventually be dominated by the term with the largest growth rate (as long as it’s coefficient $a$ is not zero). So, for deciding stability or instability, we care mainly about the eigenvalue with the largest real part, and the corresponding term of (6.10) is the one that matters.

In the two-equation case, it is convenient to represent solutions on a phase plane diagram,

---

$^8$Things get trickier when not all eigenvectors are different, but we won’t deal with that complication here.
Figure 23: Solutions near a saddle. The two eigenvalues are real and have opposite signs. The eigenvectors \( \vec{v}^{(1)}_0 \) and \( \vec{v}^{(2)}_0 \) correspond to the positive and negative eigenvalues, respectively.

with axes representing the two components of the solution (see figure 22). Because there are two equations, the eigenvalues are the roots of a quadratic polynomial. This simplifies things: either both eigenvalues are real or one is the complex conjugate of the other. The list of possible evolutionary patterns now expands to include the six cases shown in figure 22.

The first two are straightforward: both eigenvalues are real and have the same sign, so that all solutions grow (if the eigenvalues are positive) or decay (if the eigenvalues are negative). If the eigenvalues are both real but have opposite signs, the situation is more complicated (figure 23). The evolutionary pattern is called a saddle. The orientation of the saddle depends on the eigenvectors. As \( t \to \infty \), all solutions become proportional to the “growth” eigenvector, i.e. the one corresponding to the positive eigenvalue. The sign of the proportionality (the direction the solution eventually follows) depends on the initial condition. Imagine that the “decay” eigenvector (the one corresponding to the negative eigenvalue) divides the phase plane into two halves. Solutions starting on one side of the decay eigenvector go one direction, those starting on the other side go the other direction.

Limit cycles and spirals have complex conjugate eigenvalues (figure 22d,e,f). The sense of the rotation is easily determined by looking at the original equations. For example, look back to (6.2). The phase plane for this system is depicted in figure 24. For any point in the first quadrant, \( f > 0, g > 0 \), the equations tell us that \( f_x > 0 \) and \( g_x < 0 \); hence, the circulation is clockwise.
Figure 24: A limit cycle solution for the system $f_x = g, g_x = -f$.

For three or more equations, a new type of solution becomes possible: the *strange attractor* of chaotic dynamics. We will not descend into chaos here, but it’s an important topic, well worth reading up on. We close this section with an application of two-equation systems.

### 6.4 Case study: Love affairs

Any mathematician will tell you that math is not really about numbers; it’s about relationships. Here’s an example.\(^9\) Consider two people - let’s call them Romeo and Juliet - who meet and develop feelings for one another. We want to model the way those feelings evolve over time.

Let the function $R(t)$ represent Romeo’s affection for Juliet, while $J(t)$ is Juliet’s affection for Romeo. Romeo’s affection evolves according to

$$
\frac{dR}{dt} = aR + bJ. \quad (6.11)
$$

The coefficient $a$ represents Romeo’s response to his own feelings. If he is the cautious type and withdraws when he feels himself growing amorous, $a < 0$. If, on the other hand, he is an idealist who tend to love the idea of love itself, independent of the actual person, then $a > 0$. The coefficient $b$ represents his reaction to Juliet’s feelings: if her affection attracts him, then $b > 0$, whereas if he is the kind who runs from love, $b < 0$.

Juliet, of course, has her own personal way of reacting to Romeo’s feelings and to her own, and this gives a second equation analogous to (6.11). The result is the linear system

$$
\frac{d}{dt}\begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} R \\ J \end{pmatrix}. \quad (6.12)
$$

It’s easy to write down a general solution for (6.12), but it’s more interesting to look at particular cases.

**Example 15:** Suppose that our two potential lovers are out of touch with their own feelings but react strongly to each other: Romeo is drawn to affection but Juliet runs from it:

\[
\frac{dR}{dt} = aJ; \quad \frac{dJ}{dt} = -bR, \quad \text{or,} \quad A = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}
\]

where the signs are chosen so that \(a\) and \(b\) are positive. Can these two people find true love? Well, it depends on the eigenvalues. These are solutions of \(\sigma^2 + ab = 0\):

\[
\sigma = \pm i\sqrt{ab}.
\]

The eigenvalues form a complex conjugate pair with zero real part, so the solution has the form of a limit cycle (figure 22d or 24). The solution cycles through four states. For the first part of the cycle, both are in love and all is well. But soon Juliet starts to feel smothered. She suggests that they see other people. Romeo hangs in there for a while, but eventually becomes disillusioned and turns away himself. But now Juliet’s interest returns. She eventually wins Romeo back into the relationship, and the cycle begins again.

**Example 16:** Let us now suppose that Romeo and Juliet are identical in their reactions to one another:

\[
\frac{d}{dt} \begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}.
\]  

Suppose further that both are the cautious type, \(a < 0\), but that both are drawn to affection, \(b > 0\). The eigenvalues are both real, with values \(a + b\) and \(a - b\). Since \(b > 0\), the larger eigenvalue is \(a + b\). If \(b < |a|\), this is negative and the romance fizzles (figure 22a).

If \(b > |a|\), there is hope for growth, but what sort of growth? For this we need to look at the growth eigenvector, the one corresponding to the eigenvalue \(a + b\), which turns out to have \(R = J\) (figure 25). Not surprisingly given their similarities, Romeo and Juliet’s feelings for each other are the same, and they grow in intensity. The result is a fairy tale ending, with ever-growing love forever after, **provided** the initial conditions are conducive. But if the relationship gets off on the wrong
Figure 25: Saddle solution for the “birds of a feather” example with $b > |a|$. 

foot, Romeo and Juliet grow to hate each other! The dividing line is the decay eigenvector, corresponding to the second eigenvalue $a - b$. This is an example of a saddle (also see figure 22c or 23).
7 Day 8: Vector calculus

Several useful differential operations involve scalar and vector fields. A defined here, fields are simply quantities that depend on at least two spatial dimensions: $\phi = \phi(x,y,z) = \phi(\vec{x})$, or $\vec{u}(\vec{x}) = \{u(\vec{x}), v(\vec{x}), w(\vec{x})\}$.

A vector field can be differentiated in just the way you’d think: if $\vec{u}(x) = \{u(x), v(x), w(x)\}$, then
\[
\frac{d\vec{u}}{dx} = \left\{ \frac{du}{dx}, \frac{dv}{dx}, \frac{dw}{dx} \right\},
\]
and similarly for partial derivatives when the vector is a function of multiple variables. Vector derivatives can be combined with scalar derivatives in many useful ways, a few of which we’ll review here.

7.1 Scalar fields: the gradient and the directional derivative

Suppose a scalar quantity $\phi$ varies in space and (possibly) time: $\phi = \phi(x,y,z,t)$. Its differential is then given by
\[
\Delta \phi = \frac{\partial \phi}{\partial t} \Delta t + \frac{\partial \phi}{\partial x} \Delta x + \frac{\partial \phi}{\partial y} \Delta y + \frac{\partial \phi}{\partial z} \Delta z.
\]
A compact way to write this is
\[
\Delta \phi = \frac{\partial \phi}{\partial t} \Delta t + \vec{\nabla} \phi \cdot \Delta \vec{x},
\]
where $\Delta \vec{x}$ is the space increment $\{\Delta x, \Delta y, \Delta z\}$, and the vector
\[
\vec{\nabla} \phi = \left\{ \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\}
\]
is the gradient of $\phi$. So for example the time derivative of $\phi$ measured at a point moving through space, $\vec{x} = \vec{x}(t)$ is
\[
\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \vec{\nabla} \phi \cdot \frac{d\vec{x}}{dt}.
\]
This is the directional derivative. The derivative $d\vec{x}/dt$ is, of course, just the velocity of the moving point. Figure 26 shows isobars of surface atmospheric pressure versus longitude and latitude, with a few representative vectors to indicate the gradient.
Figure 26: Contours of surface atmospheric pressure as a function of longitude and latitude. "L" and "H" indicate low and high extrema, resp. Arrows show the gradient.

7.2 Divergence and curl

Two other useful vector derivatives we’ll mention here are the divergence and the curl. Unlike the gradient, these both operate on vectors rather than scalars. The divergence of a vector field \( \{u, v, w\} = \vec{u}(x, y, z) \) is

\[
\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.
\]

while the curl is given by

\[
\nabla \times \vec{u} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
u & v & w
\end{vmatrix} = \hat{i}(w_y - v_z) - \hat{j}(w_x - u_z) + \hat{k}(v_x - u_y).
\]

In a horizontal flow such that shown in figure 27, which depicts a river seen from above, the curl is purely vertical and represents the rate at which a small cross floating in the water would rotate\(^\text{10}\). In this case the curl changes sign at the velocity maximum. Note that the divergence is a scalar while the curl is a vector.

\(^{10}\)The curl is actually one-half the angular velocity.
**Figure 27:** Schematic of a river seen from above. Flow velocity is a maximum at the center and drops to near zero at the banks. Small crosses floating in the water rotate at an angular rate given by (twice) the curl of the velocity field.

**Figure 28:** A schematic showing convergence of the wind field at the surface and divergence aloft. The net divergence is nearly zero, so the flow along the axis must be vertical, leading to precipitation.
Example 17: The horizontal components of the wind field show in figure 28 can be modeled by:
\begin{align*}
u &= -\frac{1}{2}\Omega y - \alpha(H - 2z)x \\
v &= \frac{1}{2}\Omega x - \alpha(H - 2z)y
\end{align*}
with $\Omega$ twice the angular rotation rate, $\alpha$ a measure of horizontal divergence and $H$ the height of the planetary boundary layer. The horizontal divergence $u_x + v_y$ is equal to $-2\alpha(H - 2z)$. This is negative at the surface (indicating convergence) and positive at $z = H$ (indicating divergence), as shown on figure 28. As is typical in geophysical fluids, the net divergence is approximately zero:
$$\nabla \cdot \vec{u} = u_x + v_y + w_z = 0.$$ Substituting and using the condition $w = 0$ at $z = 0$, we can solve for $w$:
$$w = 2\alpha z(H - z).$$ Vertical velocity thus depends on height only, is positive and is a maximum at the center of the boundary layer. This describes a broad region of upward flow and subsequent participation (since air loses its capacity to carry water vapor as the ambient pressure decreases).

The curl of the wind field is $\nabla \times \vec{u} = \{-2\alpha H y, -2\alpha H z, \Omega\}$ (try it!). The vertical component quantifies the rotation about the low pressure center shown in figure 28. Check that the right-hand rule is obeyed. The horizontal components are less important here: they reflect the slow rotational motion inherent in the converging and diverging motions at the surface and high in the boundary layer.

7.3 The Laplacian

The Laplacian is a multidimensional extension of the second derivative, and is defined as the divergence of the gradient:
$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$ The Laplacian plays a central role in some of the most fundamental equations of applied mathematics, as we’ll see shortly. Note that the Laplacian is a scalar.
7.4 Identities of vector calculus

The following table summarizes the important properties of the operations discussed in this section. All quantities are assumed to depend on (at least) \( x, y \) and \( z \).

<table>
<thead>
<tr>
<th>name</th>
<th>input</th>
<th>formula</th>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>gradient</td>
<td>scalar ( \phi )</td>
<td>( \nabla \phi = { \phi_x, \phi_y, \phi_z } )</td>
<td>vector</td>
</tr>
<tr>
<td>divergence</td>
<td>vector ( \vec{u} = {u,v,w} )</td>
<td>( \nabla \cdot \vec{u} = u_x + v_y + w_z )</td>
<td>scalar</td>
</tr>
<tr>
<td>curl</td>
<td>vector ( \vec{u} = {u,v,w} )</td>
<td>( \nabla \times \vec{u} = {w_y - v_z, u_z - w_x, v_x - u_y} )</td>
<td>vector</td>
</tr>
<tr>
<td>Laplacian</td>
<td>scalar ( \phi )</td>
<td>( \nabla^2 \phi = \phi_{xx} + \phi_{yy} + \phi_{zz} )</td>
<td>scalar</td>
</tr>
</tbody>
</table>

Many useful identities can be shown to hold between arbitrary vector and scalar functions in three-dimensional space. Here are some examples:

1. \( \nabla \cdot (\nabla \times \vec{u}) = 0 \)
2. \( \nabla \times (\nabla \phi) = 0 \)
3. \( \nabla \times (\nabla \times \vec{u}) = \nabla (\nabla \cdot \vec{u}) - \nabla^2 \vec{u} \)
4. \( \nabla (\phi \psi) = \psi \nabla \phi + \phi \nabla \psi \)
5. \( \nabla \cdot (\phi \vec{u}) = \vec{u} \cdot \nabla \phi + \phi \nabla \cdot \vec{u} \)
6. \( \nabla \times (\phi \vec{u}) = \nabla \phi \times \vec{u} + \phi \nabla \times \vec{u} \)
7. \( \nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v}) \)
8. \( \nabla^2 (\phi \psi) = \psi \nabla^2 \phi + \phi \nabla^2 \psi + 2 \nabla \phi \cdot \nabla \psi \)
8 Day 9: PDEs revisited, special functions

In our previous discussion of PDEs we confined ourselves to functions of a single spatial dimension (and time). With the tools of vector calculus outlined in the last few subsections, we are ready to look at PDEs in two or three dimensions. As in the simpler cases, we use separation of variables to break these PDEs down into multiple ODEs, one for each spatial coordinate and one for time.

We will move from Cartesian to curvilinear coordinate systems (figure 29), and will as a result encounter some of the classical special functions of mathematical physics. These arise naturally as solutions of ODEs, just like the exponential and trig functions that show up in Cartesian coordinates. Like those simpler functions, they are “special” only in the sense that they are useful enough to be worth naming.

Perhaps the most common multidimensional PDE is Laplace’s equation $\nabla^2 f = 0$. As a simple example, a fabric stretched between fixed points, such as a tent, obeys Laplace’s equation in the form

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

where $f(x,y)$ is the elevation of the tent.
Both the diffusion equation and the wave equation involve second spatial derivatives, and both have three-dimensional extensions that use the Laplacian:

\[ \frac{\partial f}{\partial t} = \kappa \nabla^2 f \]

and

\[ \frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f. \]

Both the diffusion equation and the wave equation can be separated into a time part and a space part, \( f(\vec{x}, t) = F(\vec{x})T(t) \). For the diffusion equation, we get:

\[ \frac{1}{T} \frac{dT}{dt} = \frac{\kappa}{F} \nabla^2 F = \gamma, \]

where \( \gamma \) is the separation constant. The spatial equation, when coupled with appropriate boundary conditions, is just a differential eigenvalue problem for the eigenfunctions of the Laplacian:

\[ \nabla^2 F = \frac{\gamma}{\kappa} F, \]

where \( \gamma/\kappa \) is the eigenvalue. For the wave equation, the time part is different but the space part has the same form:

\[ \nabla^2 F = \frac{\gamma}{c^2} F. \]

For each eigenvalue, there is an eigenfunction \( F \), and the general solution is a sum (or possibly an integral) over these. When the coordinate system is Cartesian, the eigenfunctions of the Laplacian are sines and cosines, which is the fundamental reason those functions are so important.

When the coordinate system is not Cartesian, solutions are obtained in a very similar way, but the eigenfunctions of the Laplacian are no longer sines and cosines, they are Bessel and Legendre functions.

**Example 18**: Waves on a circular basin are usefully described by the wave equation in two-dimensional cylindrical (or polar) coordinates:

\[ \frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f = c^2 \left\{ \frac{1}{r} (rf_r)_r + \frac{1}{r^2} f_{\theta\theta} \right\}. \]
Here \( f(r, \phi, t) \) is the wave height at any point, \( r \) is the radial coordinate (the distance from some central point) and \( \theta \) is the azimuthal angle.\(^{11}\) Now assume a separable solution \( f(r, \theta, t) = R(r)P(\theta)T(t) \):

\[
\frac{1}{c^2} \frac{T_{tt}}{T} = \frac{(rR_r)_r}{rR} + \frac{P_{\theta\theta}}{r^2P} = -k^2.
\]

The left-hand side is purely a function of time while the right-hand side depends on space only, so they must both equal a constant. The solution of the time part is periodic, which makes sense for waves, provided the constant is negative. Accordingly, we write the constant as \(-k^2\).

Next we separate the spatial part. A simple rearrangement gives:

\[
r \frac{(rR_r)_r}{R} + k^2 r^2 = - \frac{P_{\theta\theta}}{P} = n^2.
\]

Again, the form of the separation constant has benefited from our ability to see the future. The azimuthal equation must have periodic solutions with period \(2\pi\), i.e. the solution at \( \theta = 2\pi \) must be identical to the solution at \( \theta = 0 \), which only happens if the separation constant is a squared integer (try it!).

The radial equation can now be rearranged a little to give

\[
r^2 R_{rr} + r R_r + (k^2 r^2 - n^2) R = 0.
\]

This is Bessel’s equation. The solutions are called Bessel functions, and are defined for each value of \( n \): \( J_n(kr) \) and \( Y_n(kr) \) (figure 30). The properties of the Bessel functions have been worked out using the same approach we used in section 3.9 to study exponentials and trig functions. It will come as no surprise that Bessel’s equation often arises when working in cylindrical geometries, e.g. in the study of tornadoes and other vortices as well as engineering applications like flow in a pipe.

Here are two other classical differential equations that are important enough to be named.

*Legendre’s equation*: \((1-x^2)f'' - 2xf' + n(n-1)f = 0\). Solutions are the Legendre functions \( P_n(x) \) and \( Q_n(x) \). When \( n \) is an even integer these are polynomials. Legendre functions arise naturally in spherical geometry, e.g. in global climate models.

\(^{11}\)The Laplacian and other vector differential operators are difficult to work out in curvilinear coordinates, but you can look them up in any pertinent reference book.
Hermite’s equation: \( f'' - 2xf' + nf = 0 \). These arise when studying flow near the equator. The solutions are called Hermite functions, and are polynomials when \( n \) is an even integer.
9 Exercises

Exercises for section 2: Waves and tides

1. Check (2.1) on your calculator. Pick any values you like for $a$ and $b$.

2. Spring tides are periods when the lunar and solar tides add together to create exceptionally high and low tides. During neap tides, sun and moon cancel partly each other out, creating weak tides. Here you will estimate the interval between spring (or neap) tides. The lunar tide has period 12.4hr., while the solar tide has period very close to 12hr. As a first approximation, assume the tides have equal amplitude, so that they add as in (2.1). Compute the beat frequency.

The tides at Newport for the period of this class are shown in figure 31. Although the physics of tides is complicated hugely by continents and variations in ocean depth, the period you have estimated for the spring-neap cycle is about right.

3. Storms at sea create swell with periods ranging between 10 and 20s. Suppose two storms create wavetrains with periods 17s. and 19s and equal amplitude. What is the period of the beats? How many waves arrive between the waves of largest amplitude (the "sneaker" waves)?

4. Define the mean frequency of two wavetrains as

$$f_m = \frac{f_1 + f_2}{2} \quad (9.1)$$

and the beat frequency as

$$f_b = f_1 - f_2, \quad (9.2)$$

where we assume that $f_1$ is the higher frequency. Solve (9.1,9.2) to give $f_1$ and $f_2$ in terms of $f_m$ and $f_b$.

The next time you are at the beach, try measuring $f_m$ and $f_b$. Use the formulae you developed here to determine the frequencies of the individual swells that create the beats.
Figure 31: Tides at Newport for September 12-20 2009. Note the dominant period of about two cycles per day. The slow change in the character of the oscillation is due to the spring-neap cycle, with spring tides in the final few days.

5. Trigonometric functions can also describe oscillations in space (think of ripple marks in sand as an example). Suppose an oscillation is described by \( h = h_0 \sin(kx) \) where \( k \) is a constant called the wavenumber. The wavelength \( \lambda \) is the value of \( x \) that makes the argument to the sine function (the quantity in brackets) equal to \( 2\pi \). Give an expression for \( \lambda \) in terms of the wavenumber.
Exercises for section 3.1: The derivative

1. Using (3.1) as in Example 2, demonstrate that the power rule (3.2) works for the following cases:
   (a) \( f(x) = x^4 \)
   (b) \( f(x) = x^{-2} \).

2. Using (3.2), differentiate the following functions:
   (a) \( f(x) = x^5 \)
   (b) \( f(x) = x^{12} \)
   (c) \( f(x) = x^{-3} \)
   (d) \( f(x) = x^0 = 1 \)

3. Using the differentiation rules (3.2 - 3.6) as required, differentiate the following functions. In each case, state which rules you used and how.
   (a) \( f(x) = x^5 + 2 \)
   (b) \( f(x) = 2x^{-2} \)
   (c) \( f(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \).

4. Differentiate \( f(x) = x^9 \).

5. Differentiate \( f(x) = x^9 \) by writing it as \( f(x) = x^4x^5 \) and applying the product rule. Verify that your answer matches the previous problem.

Exercises for section 3.2: Composite functions and the chain rule

1. Differentiate \( f(x) = x^6 \) in two ways. First do it directly using the power rule. Second, write \( f(x) \) as the composite function \( f(u) = u^2; \ u(x) = x^3 \) and find \( df/dx \) using the chain rule. Check that your answers match.

2. Repeat the previous Exercise, but now write \( f(x) = x^6 \) as \( f(v) = v^3; \ v(x) = x^2 \). Again, check that the answer matches.
3. You go to sea to measure the gradient of sea-surface temperature. After leaving the harbor, you lower a thermometer into the water and it reads 10°C. You then travel for 2 hours at 10 km/hr and lower the thermometer again. This time you get 15°C. Call the temperature \( T \), your distance \( x \) and the time \( t \). Using the chain rule, compute an average value of \( dT/dx \) over the distance you covered. [Hint: What you actually measure from the boat is \( T = T(t) \). The rate of change of the temperature you measure with respect to time is \( dT/dt = dT/dx \times dx/dt \). Substitute as appropriate and solve for \( dT/dx \).]

4. How is the derivative of the multiplicative inverse of a function, \( 1/f(x) \), related to the derivative of the function itself? Answer this in two ways:
   
   (a) Define \( g(f) = 1/f \) and use the chain rule to compute \( g'(x) \).
   
   (b) Write \( f(x)g(x) = 1 \), differentiate both sides with respect to \( x \) using the product rule, and solve for \( g'(x) \).

5. Consider two functions of the same independent variable, \( u(x) \) and \( v(x) \). Use the product rule and the chain rule to derive the \textit{quotient rule} for \( (u/v)' \) in terms of the individual derivatives \( u' \) and \( v' \).

**Exercises for section 3.3: Higher-order derivatives**

1. Let \( f(x) = x^4 \). Compute \( f'(x) \), \( f''(x) \), and all the remaining nonzero derivatives of \( f(x) \).

2. Again let \( f(x) = x^4 \). Show that \( f^{(4)}(x) = 4! \), where the \textit{factorial function} is defined for any integer \( n \) by

\[
    n! = n(n-1)(n-2)\cdots1.
\]

3. The position of an object is given by \( x(t) = 4t^3 \). Compute the velocity and the acceleration.

4. Compute the radius of curvature of the function \( f(x) = x^2 \) at \( x = 0 \). Sketch a graph of the function and the tangent circle to show that your answer is plausible. (Sketch the graph with a pencil. Make it clear and precise enough to convince me that your answer is probably right, but don’t fuss over it.)
5. Identify a minimum, a maximum and an inflection point of the function \( f(x) = \frac{1}{3}x^3 - x \). Sketch a graph to illustrate your result.

6. Of all rectangles with a given perimeter \( P \), show that the square has the greatest area.

7. In the early 1970s, a sociology book called “Future Shock”, by Alvin Toffler, was extremely influential in the highest levels of government. Even President Nixon was reputed to be a fan. Toffler’s thesis was that, not only is the world changing, but the rate of change is itself continually increasing. He spent many, many pages laboring to explain this complex idea. Can you phrase it concisely in terms of derivatives?

8. Consider once again the function \( f(x) = x^4 \). Compute its first four derivatives at the point \( x = 0 \). Could you characterize this point as a maximum or a minimum of \( f(x) \)? How? [Hint: A sketch will help.]

---

**Exercises for section 3.4: Taylor series**

1. Compute the first three terms in the Taylor series expansion of \( f(x) = (1 + x)^{1/2} \) about \( x = 0 \).

2. Write down the full Taylor series for \( f(x) = (1 - x^2)^{-1} \). Feel free to make use of the results from Example 4.

3. Using a calculator, evaluate your Taylor series from the previous problem at \( x = 1/2 \), one term at a time, until the result converges to three decimal places. Show the value of each term and the cumulative sum after adding each term.
Exercises for section 3.5: Approximating the derivative

1. The table below has columns for values of $x$, the function $f = x^3$, the derivative (computed analytically), the forward difference approximation to the derivative, and the centered difference approximation to the derivative. Fill in the blanks. (Ignore blanks marked “X”.)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^3$</th>
<th>$f'(x)$</th>
<th>$f'(x)$ FD</th>
<th>$f'(x)$ CD</th>
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<td>X</td>
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</tbody>
</table>

2. A typical motorcycle can decelerate from 100km/hr to zero in 40m. Here you will compute that deceleration.

   (a) Begin by using the chain rule, with $v = v(x(t))$, to show that
   
   $$ a = \frac{dv}{dt} = v \frac{dv}{dx}. $$

   (b) Now use the product rule to show that
   
   $$ v \frac{dv}{dx} = \frac{d}{dx} \left( \frac{v^2}{2} \right), \text{ hence } a = \frac{d}{dx} \left( \frac{v^2}{2} \right). $$

   (c) Approximate this derivative as a forward difference over length $L$, with initial velocity $v_i$ and final velocity $v_f = 0$, showing that the mean acceleration is
   
   $$ a = -\frac{v_i^2}{2L}. $$

   (d) Now compute the deceleration of the motorcycle. Express this deceleration in “gees”, i.e. as a multiple of $g = 9.8m/s^2$, the acceleration due to gravity.

   (e) Finally, suppose the motorcycle collides with a large, immovable object. Compute the deceleration in gees if the driver is

   i. wearing a seatbelt (deceleration distance = half bike length = 1m)

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ii. not wearing a seatbelt (deceleration distance = nose length = 2 cm).

(Problem courtesy of COAS professor/biker Bob Miller.)

Exercises for section 3.6: Antiderivatives and integrals

1. Write antiderivatives for the following:
   
   (a) \( f(x) = \frac{1}{4}x^5 \)

   (b) \( f(x) = x^{-3} + 1 + x^{12} \)

   (c) \( f(x) = (1 - x)^{-2} \)

   (d) \( f(x) = (1 - x)^{-3} \)

2. Evaluate the following:

   (a) \( \int_{1}^{2} \frac{1}{4}x^5 \, dx \)

   (b) \( \int_{2}^{3} (1 - x)^{-2} \, dx \)

   (c) \( \int_{3}^{4} (1 - x)^{-2} \, dx \)

   (d) \( \int_{2}^{4} (1 - x)^{-2} \, dx \)

Exercises for section 3.8: Differential equations

1. Suppose you fall out of an airplane starting at height \( z_0 \) so that your falling velocity is given by \( -gt \) where \( g \) is a constant, negative because your height is decreasing in time. Your height \( z(t) \) is a solution of the differential equation

\[
\frac{dz}{dt} = -gt
\]

with the initial condition

\( z(0) = z_0. \)
Derive the solution

\[ z(t) = a + bt^2, \]

and identify the constants \( a \) and \( b \).

2. Classify the following ODEs in terms of order, linearity and homogeneity.

(a) \( f'' + 3f = 0 \)
(b) \( f' + 3f^2 = 0 \)
(c) \( gg'' + g = 4 \)
(d) \( fg'' + g = 4 \), where \( g \) is the solution to be determined and \( f \) is a given function of \( x \).

3. Consider the equation \( f'' - 2x^{-2}f = 0 \).

(a) What is the order of this equation?
(b) Is it linear or nonlinear? Homogenous or inhomogeneous?
(c) Substitute \( f = x^\alpha \), where \( \alpha \) is a constant to be determined, and thereby show that \( x^2 \) and \( x^{-1} \) are both solutions.
(d) Show that the linear combination \( ax^2 + bx^{-1} \), where \( a \) and \( b \) are constants, is also a solution.

Exercises for section 3.9: Exponential functions

1. Consider the problem of compound interest. Suppose we invest one dollar at interest rate \( r \) for a time \( t = 1 \) year. At the end of the year, the interest is paid and the principal becomes \( P = 1 + rt \) dollars. In this case we say the interest is compounded annually. If the interest is compounded semi-annually, then the principal after one year is \( P = (1 + \frac{rt}{2})^2 \). In general, if interest is compounded \( N \) times per year, the principal after any time \( t \) is \( P_N(t) = (1 + \frac{rt}{N})^N \).

(a) Compute the derivative \( P_N'(t) \).
(b) Compute the ratio \( \frac{P_N'}{P_N} \).
(c) Compute the limit of $\frac{P'_N}{P_N}$ as $N \to \infty$. (In this limit we say the interest is compounded continuously.)

(d) Now consider the function $E(t) = \lim_{N \to \infty} (1 + \frac{t}{N})^N$. What can you say about the relation between $E$ and its derivative?

2. Carbon found in living creatures is a blend of two isotopes: $C_{12}$, which is stable, and $C_{14}$, which is radioactive. $C_{14}$ decays according to the exponential decay law

$$\frac{dC_{14}}{dt} = -rC_{14} \quad (9.3)$$

where $r$ is the decay rate, or the inverse of the average lifetime of a $C_{14}$ molecule, equal to about $1/(8000 \text{ years})$. As long as a creature is alive, $C_{14}$ is continually replenished, but at death the fraction of $C_{14}$ begins to decrease. A measurement of that fraction therefore reveals the time elapsed since the creature died. The smallest fraction of $C_{14}$ that can be accurately measured is about 0.001, or 0.1%. The corresponding age is the greatest than can be measured - any material that has a smaller fraction of $C_{14}$ than 0.001 can only be characterized as, in technical terms, “really old”. What is this maximum age? [Hint: You may find (3.23) useful in solving (9.3).]

3. Consider the following equation and boundary conditions:

$$f''(x) + f(x) = 0; \quad (9.4)$$

$$f(0) = 0; \quad f'(0) = 1. \quad (9.5)$$

In this exercise you will study the solutions of this problem much as we studied the exponential solutions of (3.18) previously.

(a) Suppose that (9.4) has a solution, which we’ll call $S(x)$. Differentiate (9.4) to show that the derivative $S'(x)$ must also be a solution. We’ll call this second solution $C(x)$.

(b) You know that $S' = C$; now show from (9.4) that $C' = -S$.

(c) Using the chain rule, show that $\frac{1}{2} f'^2 = ff'$ for any function $f(x)$.

(d) Substitute the solution $S(x)$ into (9.4), then multiply through by $S'$. Using your result from 3c and (9.5), show that $S^2 + C^2 = 1$ for all $x$.

(e) To be able to satisfy most boundary conditions, the two solutions $S$ and $C$ must be \textit{linearly independent}, i.e. one cannot simply be a multiple of the other. Show that
S and C are linearly independent by showing that there is no constant α such that 

$S = αC$ and $S' = αC'$.

(f) You’ve shown that $S' = C$. Now show that $S'' = -S$, $S''' = -C$, $S'''' = S$, and all higher derivatives follow the same cycle.

(g) Using your result from 3f, show that the Taylor series expansion for S about $x = 0$ is

$$S(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots.$$  

(9.6)

(h) Differentiate (9.6) to obtain the Taylor series expansion for C about $x = 0$.

(i) Is $S$ even [$S(-x) = S(x)$] or odd [$S(-x) = -S(x)$]? Is $C(x)$ even or odd?

(j) Write out the Taylor series expansion for $S(x+y)$ about $x$, using what you know to simplify all derivatives. Collect terms proportional to $S(x)$ and to $C(x)$, then use your previous results to derive the addition rule $S(x+y) = S(x)C(y) + C(x)S(y)$.

(k) In the addition rule you just derived, replace $y$ with $-y$ to obtain the subtraction rule.

(l) Combine the addition and subtraction rules to obtain the product rule:

$$S(x)C(y) = \frac{1}{2}[S(x+y) + S(x-y)].$$  

(9.7)

I trust you recognize what $S$ and $C$ stand for...? Note that (9.7) is just (2.1) with $a = x+y$ and $b = x-y$. We’ll stop here, but it should be evident that the whole science of trigonometry is contained in (9.4), and can be extracted without ever looking at a triangle.
Exercises for section 3.10: Complex exponential, trigonometric and hyperbolic functions

1. Using (3.32), derive Euler’s famous relationship between $e$, $i$ and $\pi$: $e^{i\pi} = -1$.

2. Use the relation (3.32) to derive the addition formula for the sine function: $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$. [Hint: First apply the summation property of the exponential function to get $e^{i(x+y)} = e^{ix}e^{iy}$. Then expand both sides using (3.32) and equate the imaginary parts.]

3. Derive the corresponding identity for $\cos(x + y)$.

4. Derive the first two nonzero terms in the Taylor series expansion for $\tan(x)$ about $x = 0$.

5. Differentiate the definitions of the sinh and cosh functions to show that

$$\frac{d}{dx} \sinh(x) = \cosh(x); \quad \frac{d}{dx} \cosh(x) = \sinh(x).$$

How do these differ from the corresponding relationships between sin and cos and their derivatives?

6. Derive the first three nonzero terms in the Taylor series expansion for $\cosh(x)$ about $x = 0$.

7. Are the sinh and cosh functions even or odd?

8. Using (3.32) and the definitions of the sinh and cosh functions, show that $\sinh(ix) = i\sin(x)$ and $\cosh(ix) = \cos(x)$.

9. In your result from the previous problem, make the substitution $x = iy$, and thereby show that $\sin(iy) = i\sinh(y)$ and $\cos(iy) = \cosh(y)$. [Hint: Your results from two problems ago on the parity of sinh and cosh may be useful here.]

Exercise for section 3.11: Newton’s 2nd law for small oscillations

The density (mass per unit volume) of seawater is generally a function of the vertical coordinate: $\rho = \rho_W(z)$. An object with density $\rho_0$ (a fish egg, say) tends
to sink or float until it finds its equilibrium depth, the depth at which $\rho_W = \rho_0$. If the object should be displaced slightly from the equilibrium depth, it will bob up and down. Suppose that the volume of the object is $V$, so that the mass is $\rho_0 V$. The force of gravity acts downward with strength $\rho_0 V g$, while buoyancy acts upward with strength $\rho_w V g$ (the weight of the displaced water). If $z$ is the object’s vertical displacement (its position, measured from the equilibrium depth), then Newton’s second law $ma = F$ gives:

$$\rho_0 V \frac{d^2z}{dt^2} = \rho_w(z) V g - \rho_0 V g.$$  \hfill (9.8)

If the displacement $z$ is small, the water density at $z$ can be approximated by the first two terms of its Taylor series: $\rho_w(z) = \rho_0 + \rho_w'(0) z$.

1. Substitute this into (9.8), and solve the resulting equation for $z(t)$, keeping in mind that $\rho'_w(0)$ is negative, since density decreases towards the surface.

2. Show that the solution represents an oscillation whose angular frequency is given by

$$N = \sqrt{-\frac{g \rho'}{\rho_0}}.$$

$N$ is called the *buoyancy frequency*, and is a common measure of density stratification in geophysical fluids.
Exercise for section 3.12: Differential eigenvalue problems

1. Give the eigenvalues and corresponding eigenfunctions for the following:

\[ f''(x) + \lambda^2 f(x) = 0; \]
\[ f(0) = 0; \quad f(1) = 0. \]

2. Internal waves are supported by density differences in the ocean or atmosphere in the same way that surface waves are supported by the density difference between water and air. Like surface waves, internal waves have a frequency and a horizontal wavelength, but they also have a vertical structure. For one important class of internal waves, the vertical structure is described by the following differential eigenvalue problem:

\[ \phi''(z) + \frac{N^2}{c^2} \phi(z) = 0; \]
\[ \theta(0) = 0; \quad \theta(-H) = 0. \]

Here, \( \theta \) is the amplitude of the vertical displacement of the fluid due to the wave, which varies with depth. \( N \) is a measure of density variations that you will derive elsewhere, and \( c \) is the wave speed. The ocean surface is at \( z = 0 \), and the bottom is \( z = -H \). \( N \) and \( H \) are taken to be constants. The solutions of (9.9) are called hydrostatic normal modes, or just modes.

(a) Write down the solutions of this eigenvalue problem, including expressions for \( \theta_n(z) \) and \( c_n \) where \( n = 1, 2, \cdots \) is the mode number.

(b) Sketch \( \theta(z) \) for mode 1 and mode 2.

(c) Describe in words the difference between the vertical displacement patterns of mode 1 and mode 2.
Exercises for section 4.1: Multivariate functions and partial derivatives

1. Compute \( f_x \) and \( f_t \) using (4.1):
   \[(a) \quad f(x,t) = x^3 - t^2 \]
   \[(b) \quad f(x,t) = x^2t^2 \]

2. Compute the partial derivatives using any method you like. Also give an expression for the isocontour slope (the derivative of the second argument with respect to the first).
   \[(a) \quad f(t,x) = 1/(t^2 + x^3) \]
   \[(b) \quad f(x,z) = 3 + 4x + \frac{1}{4}x^4z^{-4} \]
   \[(c) \quad f(x,y) = x^2/(1 + y^2) \]

3. In the following, \( V \) is a constant. Compute \( f_x \), \( f_t \), and the isocontour slope \( dx/dt \).
   \[(a) \quad f(t,x) = x - Vt \]
   \[(b) \quad f(t,x) = \sin(x - Vt) \]
   \[(c) \quad f(t,x) = 1/(x - Vt)^2 \]

4. In a certain region of the ocean, the sea surface is warmer to the east: \( T_x = 0.1 \)C/km. After sunrise, it also warms due to solar heating: \( T_t = 0.001 \)C/s. You measure the temperature from a boat traveling to the east at gradually decreasing velocity: \( V = V_0 - at \), where \( V_0 = 10 \)m/s and \( a = 0.01 \)m/s\(^2\). Measurement is continued until the boat comes to rest.
   \[(a) \quad \text{How long does it take the boat to come to rest?} \]
   \[(b) \quad \text{Sketch a graph of the boat’s trajectory on the } x-t \text{ plane.} \]
   \[(c) \quad \text{Compute the measured temperature as a function of time.} \]
   \[(d) \quad \text{Sketch a plot of the measured temperature versus time.} \]
   \[(e) \quad \text{Suppose you knew the boat’s trajectory } x(t) \text{ and the measured temperature } T(t), \]
   \[\text{but you did not know } T_x \text{ and } T_t. \text{ Would you be able to figure out } T_t \text{ and/or } T_t? \]
Exercises for section 4.3: Solution of PDEs by partial integration

1. Let $f(x,t) = 2x$. Find the general solution using partial integration.

2. Let $f(x,t) = 2t$. Find the general solution using partial integration.

3. The pressure in the ocean is $p(x,y,z,t)$. At the ocean surface, $p$ is just the atmospheric pressure $p(x,y,0,t) = p_a(x,y,t)$, which we’ll assume is known. (By definition, $z = 0$ at the surface and $z < 0$ in the interior.) We also assume that, in the ocean interior, the pressure is hydrostatic:

$$p_z = -\rho g$$

where $\rho$ is the density of seawater, assumed uniform, and $g = 9.81\text{m/s}^2$ is the acceleration due to gravity. Using partial integration, find the complete solution for $p(x,y,z,t)$ in terms of $\rho, g$ and $p_a$.

Exercise for section 4.4: Solution of PDEs by separation of variables

1. Write out the Taylor series expansions for an arbitrary function $f(x)$ about some point $x_0$, and for another function $g(y)$ about $y_0$. Under what conditions can these two expressions can be equal for all $x$ and $y$?

2. Check by substitution that (4.13) satisfies (4.6), (4.7) and (4.8) with $T_0$ given by (4.12).

Exercises for section 4.5: Fourier series

Figure 33 shows a string under tension between two fixed points, like on a guitar or a violin. Suppose the string is stretched initially to a shape $h_0(x)$ then released. Subsequent motion is governed by the one-dimensional wave equation:

$$h_{tt} = c^2 h_{xx},$$
where the constant $c$ is the phase speed of waves on the string. The boundary conditions are

$$h(0,t) = 0; \quad h(L,t) = 0.$$ 

The string is initially stationary, so the initial condition is

$$h_t(x,0) = 0,$$

along with

$$h(x,0) = h_0(x).$$

1. Solve the problem via the same sequence of steps as was followed in sections 4.4 and 4.5 for the temperature in a bar. (You don’t have to work out the coefficients in the Fourier series.)

2. What is the main difference between the time dependence of your solution and the time dependence of the temperature in a bar? How do you interpret this difference physically?

3. Describe the way the time dependence of a given term in the Fourier series depends on the length scale associated with that term.
Exercises for section 5.1: Vectors and matrices

1. Let

\[
\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}; \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad \vec{w} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}; \quad \vec{x} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}
\]

Compute the following:

(a) \(\vec{u} + \vec{v}\)
(b) \(2\vec{u}\)
(c) \(\vec{u} \cdot \vec{v}\)
(d) \(\vec{v} \cdot \vec{w}\)
(e) \(\vec{w} \cdot \vec{x}\)
(f) \(|\vec{u}|\)
(g) \(|\vec{v}|\)
(h) \(|\vec{w}|\)
(i) \(|\vec{x}|\)
(j) \(\vec{u} \times \vec{v}\)
(k) component of \(\vec{u}\) in the direction of \(\vec{v}\).
(l) component of \(\vec{v}\) in the direction of \(\vec{u}\).
(m) component of \(\vec{w}\) in the direction of \(\vec{x}\).
(n) component of \(\vec{x}\) in the direction of \(\vec{w}\).

2. Verify the relations (5.1) - (5.4) above using the given vectors \(\vec{u}, \vec{v}, \vec{w}\) and \(\vec{x}\).

3. (a) Is a diagonal matrix necessarily symmetric?
   (b) Is a symmetric matrix necessarily diagonal?

4. Which of the following matrix equations is consistent?
   (a) \(A_{ij}B_j = \Gamma_i\)
   (b) \(A_{ij}B_i = \Gamma_i\)
   (c) \(A_{ij}B_i = \Gamma_j\)
Exercises for section 5.2: Cartesian position vectors

1. Evaluate the following products:

   (a) $\hat{i} \cdot \hat{j}$
   (b) $\hat{i} \cdot \hat{i}$
   (c) $\hat{i} \cdot \hat{k}$
   (d) $\hat{j} \cdot \hat{k}$
   (e) $\hat{i} \times \hat{i}$
   (f) $\hat{i} \times \hat{j}$
   (g) $\hat{j} \times \hat{k}$
   (h) $\hat{k} \times \hat{i}$
   (i) $\hat{j} \times \hat{i}$
   (j) $\hat{k} \times \hat{j}$
   (k) $\hat{i} \times \hat{k}$
   (l) $\vec{p} \cdot \hat{j}$
   (m) $\vec{p} \times \hat{j}$

2. Imagine an east-north-up coordinate system with origin located in our classroom, and a position vector indicating the highest point on the Yaquina Bay bridge. Estimate the values of the $x$, $y$ and $z$ coordinates in meters.

Exercises for section 5.3: Matrices as transformations

1. Which of the following expressions are equivalent to $A_{ij}B_{jk}$?

   (a) $A_{im}B_{mk}$
   (b) $A_{mi}^TB_{mk}$
   (c) $B_{jk}A_{ij}$
2. For each matrix, compute the determinant and describe the effect that the matrix has on a general column vector.

(a) \[ \begin{bmatrix}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 3 \\
\end{bmatrix} \]

(b) \[ \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \]

(c) \[ \begin{bmatrix}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 3 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \]

**Exercises for section 5.4: Eigenvalues and eigenvectors**

Let \( A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \).

1. Compute the determinant. Is \( A \) singular?

2. Describe in words the effect the matrix has on a general column vector \( \vec{v} \).

3. Compute all eigenvalues and eigenvectors of \( A \). Guided by these results, describe in words the class (or classes) of vectors whose direction is unchanged by \( A \).
Exercises for section 6: linear systems of ODEs

1. Do opposites attract? Predict the evolution of Romeo and Juliet’s relationship if:

\[
\frac{d}{dt} \begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}. \tag{9.10}
\]

(a) Set up the equations and find the eigenvalues. What is the nature of the behavior near equilibrium for the cases \(|a| > |b|\) and \(|a| < |b|\)?

(b) For the case \(|a| > |b|\), there are four possible sub-cases \(a > 0, b > 0, a > 0, b < 0, \) etc. For each such case, find the growth and decay eigenvectors, sketch them on the \(R - J\) phase plane, and use them to sketch a few representative trajectories that the relationship could follow. Conclude with a statement of how Romeo and Juliet’s future together depends on the personality characteristics quantified by \(a\) and \(b\).

(c) For the case \(|a| < |b|\), there are once again four possible sub-cases. For each, sketch the phase plane behavior. [Hint: the direction of the trajectory is most easily determined by looking at the original equations and asking how \(R\) and \(J\) will evolve in some simple case, e.g. the quadrant \(R > 0, J > 0\).]

2. Can you think of a case where love can grow exponentially but hate cannot? What does this suggest about real human relationships?

3. Discuss strengths and weaknesses of the linear system (6.12) as a model for real personalities and relationships. How could it be improved?

4. Analyze the stability of the second equilibrium of (6.3), where \(F = r/a\) and \(H = s/b\).

5. **1D problem:** A bubble of volume \(V\) rises through a glass of beer, propelled by buoyancy and retarded by viscous friction. Its vertical position is \(z(t)\). The mass of the bubble is \(\rho_A V\), where \(\rho_A\) is the density (mass per unit volume) of air. The mass of water displaced by the bubble is \(\rho_W V g\), where \(\rho_W\) is the density of water. The resulting upward force due to buoyancy is \(\rho_W V g\), where \(g = 9.8 m/s^2\) is the acceleration due to gravity. The downward force due to friction is proportional to the square of the bubble’s velocity, \(w = dz/dt\) (see figure 34). Newton’s second law is therefore

\[
\rho_A V \frac{dw}{dt} = \rho_W V g - kw^2. \tag{9.11}
\]
(a) The bubble accelerates upward until it reaches its terminal velocity $w_0$, the velocity at which $dw/dt = 0$. Give an expression for $w_0$.

(b) suppose now that something disturbs the bubble’s motion, so that it’s velocity becomes $w_0[1 + \varepsilon(t)]$. Approximate $kw^2$ using the first two terms in the binomial expansion for $(1 + \varepsilon)^2$. Substitute the results into (9.11) and derive a differential equation for $\varepsilon$. [Hint: Note that the terms not proportional to $\varepsilon$ cancel!]

6. **3D problem:** The Lorenz equations were an early model of nonlinear convection in the atmosphere, and one of the first problems to be solved on an electronic computer. The solution was unexpectedly complex, and led to the modern science of chaos (figure 35). A physical model of this system, the "chaos waterwheel", is on display in the HMSC visitor center.

The equations are as follows:

$$\begin{align*}
\frac{dx}{dt} &= Py - Px \\
\frac{dy}{dt} &= rx - y - xz \\
\frac{dz}{dt} &= xy - bz,
\end{align*}$$

(9.12)

where $P$, $r$ and $b$ are constants.

(a) Find the values of $x$, $y$ and $z$ at all equilibrium points. (Hint: there are three.)

(b) For each equilibrium, write down the stability matrix.

(c) For the simplest of the three equilibria (it’s obvious), solve for the three eigenvalues in terms of $P$, $r$ and $b$.

(d) Show that at least one eigenvalue has positive real part as long as $r > 1$ and $P < -1$. 

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Figure 34: Forces on rising bubble restrained by friction.
Figure 35: Phase diagram showing a chaotic solution of the Lorenz equations projected onto the $x – z$ plane.

**Exercises for section 7: Vector calculus**

1. Suppose $f(x,y) = 1/(x^2 + 2y^2)$, and $f$ is measured at a point that moves in time: $x = 2t$, $y = -t$. Evaluate $df/dt$ in two ways:

(a) by direct substitution and differentiation

(b) using (7.1).

In both cases the result should be given as a function of $t$.

2. Sketch

(a) a graph showing isocontours of $f(x,y)$ and the trajectory $y = y(x)$ on the $x – y$ plane,

(b) a graph showing $f(t)$. Note that $f(t)$ is just the cross-section of $f(x,y)$ along the trajectory you sketched.

(c) a graph showing $df/dt$.

3. Suppose that $f$ in the previous problem is really the pressure $p$, i.e. $p = P/(x^2 + 2y^2)$, with $P$ constant, and the force exerted on a fluid parcel of mass $m$ by the pressure gradient is

$$m \frac{du}{dt} = -\nabla p.$$

(a) Write an expression for this force, and
(b) illustrate it on your isocontour plot from the preceding question by means of a few representative arrows.

4. Evaluate both the gradient and the Laplacian for each of the following

(a) \( f(x,y,z) = x^2 + 2y^2 + 3z^2 \)

(b) \( f(x,y,z) = 1/(x^2 + y^2 - z^2) \) [Hint: To simplify the algebra, write \( f \) as a composite function \( f = 1/g \), where \( g(x,y,z) = x^2 + y^2 - z^2 \), and use the chain rule. Look for opportunities to simplify the result.]

(c) \( f(x,y,z) = \sin(x) \cos(y) \sin(z) \)

(d) \( f(x,y,z) = \sin(kx) \cos(ly) \sinh(mz) \), where \( k, l \) and \( m \) are constants.

5. Verify each of the identities listed in section 7.4 for the case:

\[
\vec{u} = \begin{bmatrix} x \\ y^2 \\ x + z^3 \end{bmatrix}; \quad \vec{v} = \begin{bmatrix} y \\ 0 \\ x^2 \end{bmatrix}; \quad \phi = 3x + 4 \quad \psi = 2y^2 + x.
\]